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PROJECT FINAL REPORT

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[1] Usually the contact person of the coordinator as specified in Art. 8.1. of the Grant Agreement.

4.1 Final publishable summary report

4.1.A Executive summary

The Marie Curie fellowship entitled "The Circle Method, Character Sums, and Quadratic Forms," funded the research of Dr. Lillian Pierce at the University of Oxford for two years, with the mentorship of Prof. Roger Heath-Brown, FRS. During the fellowship period, Dr. Pierce engaged in research in pure mathematics on problems in analytic number theory and harmonic analysis. The results of this research will appear in several significant papers published in high-quality peer-reviewed journals; the research is also highlighted on the project webpage. Dr. Pierce has presented this research at over 30 national and international seminar and conference events. At the conclusion of the fellowship, Dr. Pierce received numerous offers of faculty positions at universities in the US and Europe. She will be a Bonn Junior Fellow (w-2 professor) at the Hausdorff Centre in Bonn in 2013, after which she will join the faculty at Duke University in the United States.

4.1.B Summary description of project context and objectives

This project in analytic number theory aims to investigate a series of open problems linked by the themes of the circle method, exponential and character sums, and quadratic forms.

A fundamental problem in analytic number theory is the determination of asymptotics for the number of solutions to a Diophantine equation $F(\mathbf{x}) = 0$, where $F(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$ is an irreducible form. The two key questions are: do any rational points lie on the hypersurface $F(\mathbf{x}) = 0$, and if so, are they equidistributed? A classical example of this type of problem is Waring's problem, which asks for the number of representations of a positive integer as a sum of s d-th powers. Many historic problems arise from interest in Diophantine equations themselves, but in fact a surprising array of problems in analytic number theory may be translated into questions about integral or rational solutions to Diophantine equations, making the ability to compute the distribution of such points an important tool.

The most powerful analytic method for counting rational points on such a hypersurface is the circle method of Hardy and Littlewood. Initially developed by Hardy and Ramanujan to study the partition function, the circle method has been honed over many decades into an effective and flexible method. However, it still has not reached its full predicted power, and this proposal aims to push certain aspects of the circle method farther toward its limits.

For a degree d homogeneous form F in s variables, heuristics predict that the classical formulation of the circle method will be effective only for s sufficiently large with respect to d, specifically $s \geq 2d + 1$. However, for general nonsingular forms, for example, the best known results due to Birch [1] allow only $s \geq 1 + (d-1)2^d$. More is known for diagonal forms, such as in Waring's problem, where the methods of many researchers, notably including Vinogradov, Wooley, and Ford [11], have brought the number of variables down to $s \sim d^2 \log d$. But a major open problem is to reduce the number of variables down to the conjectural limit $s = 2d + 1$.

At the heart of the classical circle method lies the division of the unit interval into major and minor arcs: the major arcs, centred at rational points a/q with bounded denominators $1 \leq q \leq Q$, contribute the main term in the asymptotic, while the minor arcs, forming the complement of the major arcs in the unit interval, contribute the error term. A key limitation of the method is the bound for the minor arcs, for which Hua's lemma (or for large d, the mean value estimates of Vinogradov) is used. Hua's inequality states that

$$\int_0^1 |S(\alpha)|^s d\alpha \ll_{d,\epsilon} N^{s-d+\epsilon},$$

for any $\epsilon > 0$, as soon as $s \geq 2^d$, where $S(\alpha) = \sum_{n \leq N} \exp(2\pi i \alpha n^d)$. Prof. D. R. Heath-Brown's sharpened version [15] allows the condition on s to be relaxed to $s \geq \frac{7}{8}2^d$, for $d \geq 6$, with corresponding consequences for reducing the number of variables required in Waring's problem. This proposal aims to obtain a similar improvement on Hua's inequality even for $d \geq 4$ (Objective B).

Another method for reducing the number of variables required for the successful application of the circle method is Kloosterman's refinement [23]; this technique exploits cancellation among terms related to approximating rationals a/q with different numerators a, for each fixed q. A further development formulated by Hooley, the so-called double Kloosterman refinement, further exploits averaging over the denominators q. Prof. Heath-Brown has used this formulation to prove a number of asymptotics for representations by quadratic forms [16] and (conditionally) for cubic forms [17]. The power of the technically demanding double Kloosterman refinement method remains to be further developed. One intriguing application (Objective A2) is to conditional asymptotic results for cubic forms, where one would desire to reduce the number of variables required to $2d + 1 = 7$ (currently only $s \geq 9$ is known, due to Hooley). A further proposed application (Objective A3) is to conditional results on equal sums of s k-th powers à la the work of Prof. Heath-Brown [17] (and independently Hooley [20]) on the Hypothesis K^* of Hardy and Littlewood for sums of three cubes.

Kloosterman refinements have so far been limited by the fact that no two-dimensional Kloosterman refinement (not to be confused with the double Kloosterman refinement described above) has yet been successfully developed. This would take the form of a decomposition of the unit square, rather than the unit interval, that harvests cancellation among terms associated with different two-dimensional approximating rational pairs $(a_1/q_1, a_2/q_2)$. This proposal aims to develop the first higher dimensional Kloosterman refinement (Objective A1), with an immediate application to the question of whether two given quadratic forms must simultaneously attain prime values.

When applicable, the circle method provides not only information on the number of rational solutions but also completely answers the question of equidistribution. However, for certain problems, an upper bound for the number of integral or rational points lying on an algebraic variety within a bounded region suffices. When the region is constrained by an underlying problem of number theoretic interest, counting points may become unfeasible with existing methods if the region is highly asymmetrical. One successful method for tackling precisely such difficult problems arose in the work of Lillian Pierce [29], in which integral points in asymmetrical regions on surfaces of the form $y^2 = f(x_1, x_2)$ were counted by applying a variant of Prof. Heath-Brown's square sieve [14] in combination with his q-analogue of van der Corput's method. This proposal aims to develop a generalisation (Objective C) of this method for higher dimensional problems of this kind.

Another broad area of analytic number theory relates to bounding character sums. Such bounds not only play a key role in the circle method, but also in a vast array of number theoretic problems and even in problems in harmonic analysis, such as those considered in the doctoral work of Pierce [30], where exponential sums take the place of oscillatory integrals.

There are strong classical results for complete character sums, ranging from the results of Gauss to the highly powerful Deligne bounds for exponential sums to a prime modulus. But for short character sums, much remains unknown. As an example, consider the sum

$$\sum_{M \leq n < M+N} \chi(n),$$

where χ is any nontrivial Dirichlet character to a prime modulus p. Classical methods, such as those of Pólya and Vinogradov, lead to a bound of the form $O(p^{1/2} \log p)$. But for short sums, with $N \leq \sqrt{p}$, obtaining a nontrivial bound is much more difficult. The work of Burgess goes beyond Pólya-Vinogradov, proving bounds of the form $O_r(N^{1-1/r} p^{(r+1)/(4r^2)} \log p)$, for any positive integer r. Such nontrivial bounds led to new results on the distribution of quadratic residues modulo p [4] and subconvexity bounds for Dirichlet L-functions [5], [6].

Burgess' method is extraordinarily useful, and has been further developed and effectively applied by Prof. Heath-Brown to hybrid bounds for L-functions and upper bounds for the least square-free integer in an arithmetic progression, and by Pierce [27], [28] to counting solutions to polynomial congruence relations, with applications to class numbers of quadratic fields. This proposal aims to develop Burgess' method in several new directions (Objective D).

We return to the subject of quadratic forms: although they have been studied extensively since the pioneering work of Gauss, a broad array of interesting questions remains open. Here we single out three questions of interest. We have already mentioned the question of whether two quadratic forms must simultaneously attain prime values; for general quadratic forms, no such result is known. This proposal describes a method to resolve this question (Objective A1).

Second, the well-known Oppenheim conjecture for indefinite quadratic forms, as stated by Davenport and Heilbronn and later proved by Margulis [25], states that the values of an indefinite quadratic form in n variables are dense in $\mathbb{R}$, provided that $n \geq 3$. Analogously, for positive definite quadratic forms, Davenport and Lewis [8] conjectured that the difference between successive values of such a form $Q(m)$ converges to zero as $|m| \to \infty$, as long as $n \geq 5$. This has recently been proved by Götze [12], who further conjectures that this should remain true for $n = 3, 4$. The methods developed by Götze, in particular the treatment of pairs of theta functions, raise the question of whether similar results may be obtained for simultaneous values of a pair of quadratic forms. We propose (Objective E1) to examine this method for further applications.

Third, although originally number theoretic objects, quadratic forms and their associated theta functions have also been used by Pierce [31] to answer questions in harmonic analysis. Additionally, recently Bourgain [2] has investigated the nonlinear Schrödinger equation on "irrational tori," defined in terms of diagonal quadratic forms with positive real coefficients, at least one of which is irrational. In determining the local well-posedness of the NLS, two number theoretic questions arise: counting how often a quadratic form takes values close to a prescribed value, and counting how often two quadratic forms simultaneously take values close to two prescribed values. While Bourgain is able to obtain certain bounds for these quantities, he poses the problem of obtaining sharp bounds, conjecturally depending on the Diophantine properties of the coefficients of the quadratic forms. Any improvement to these two types of bound would translate immediately into local well-posedness results for the nonlinear Schrödinger equation. We propose (Objectives E2, E3) to examine this problem for possible improvements via classical number theoretic methods.

4.1.C Description of main results

4.1.C.1 On a conjecture of Serre

We describe here the main result of Objective C, which led to a theorem that proves (indeed improves on) a conjecture of Serre, in a special case. This work appeared in the peer-reviewed paper "Counting rational points on smooth cyclic covers," Journal of Number Theory, 132 (2012), pp. 1741-1757, co-authored by Prof. D. R. Heath-Brown and Dr. L. B. Pierce, and a preprint is available on the open access repository ArXiv.

Let $F(\mathbf{x}) \in \mathbb{Z}[x_1, ..., x_n]$ be an irreducible form of degree mr with $r \geq 2$, $m \geq 1$, such that the projective hypersurface defined by $F(\mathbf{x}) = 0$ is smooth. In this paper we will investigate the number of integer solutions to

$$y^r = F(x_1, \ldots, x_n) \tag{1}$$

with $|x_i| \leq B$. Our interest stems from the fact that an upper bound for the number of such points provides an upper bound for the number of rational points on cyclic covers of $\mathbb{P}^{n-1}$.

The density of rational points on covers of projective space is the subject of a well-known conjecture of Serre. Precisely, given a finite cover $\phi : X \to \mathbb{P}^{n-1}$ over $\mathbb{Q}$, where $n \geq 2$, define the counting function

$$N_B(\phi) = \#\{P \in X(\mathbb{Q}) : H(\phi(P)) \leq B\}.$$

Here H is the usual multiplicative height function on $\mathbb{P}^{n-1}$. Using a sieve method, Serre [33] proved that there exists $\gamma < 1$ such that

$$N_B(\phi) \ll B^{(n-1)+\frac{1}{2}}(\log B)^{\gamma}, \tag{2}$$

as long as the degree of f is at least two. In fact, however, Serre conjectures that

$$N_B(\phi) \ll B^{n-1}(\log B)^{c}, \tag{3}$$

for some c, for covers of any degree $r \geq 2$ (see Theorems 3, 4 of Chapter 13 in [33]).

Several results are known in this direction. Broberg [3] has applied Heath-Brown's determinant method [18] to prove results for covers of $\mathbb{P}^1$ and $\mathbb{P}^2$. In the case of $\mathbb{P}^1$, Broberg proves that for $\phi : X \to \mathbb{P}^1$ of degree $r \geq 2$,

$$N_B(\phi) \ll_{\phi,\varepsilon} B^{2/r+\varepsilon}.$$

For $\phi : X \to \mathbb{P}^2$ of degree $r \geq 3$, Broberg proves that

$$N_B(\phi) \ll_{\phi,\varepsilon} B^{2+\varepsilon},$$

and if ϕ is of degree 2, then

$$N_B(\phi) \ll_{\phi,\varepsilon} B^{9/4+\varepsilon}.$$

These results nearly prove Serre's conjecture (3) for $n = 2, 3$.

Recently, Munshi [26] considered the case in which ϕ is a *smooth cyclic* cover of $\mathbb{P}^{n-1}$, given by an equation of the type (1) with F a nonsingular form. In this situation he proves that for all $n \geq 2$, one has

$$N_B(\phi) \ll_{\phi} B^{n-\frac{n}{n+1}}(\log B)^{\frac{n}{n+1}}. \tag{4}$$

Note that if $n \geq 2$, this improves on (2), and even approaches Serre's conjecture in the limit as $n \to \infty$.

Unpublished results of Salberger on the dimension growth conjecture [18, Conjecture 2] imply the truth of Serre's conjecture with $(\log B)^c$ replaced by B^{ε} for covers ϕ given by (1) with F a form of precisely degree r, for any $r \geq 2$. But note that while (3) is as good as one can hope for in complete generality, one might expect an estimate of the shape

$$N_B(\phi) \ll B^{n-m(r-1)}(\log B)^{c}$$

for covers of Munshi's type, under favourable circumstances. Thus Serre's conjecture probably does not reflect the whole truth in this area.

In this paper, we again start from the foundation of Munshi's approach: given a form F as above, the equation (1) defines a variety X in weighted projective space $\mathbb{P}(m, 1, \ldots, 1)$, where the first coordinate y has weight m and the coordinates $x_1, \ldots, x_n$ have weight 1. The variety X can now be regarded as a cyclic r-sheeted cover of $\mathbb{P}^{n-1}$, given explicitly by the map $\phi : X \to \mathbb{P}^{n-1}$ that takes the point $(y, x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n)$. Thus our attention turns to counting perfect r-th power values of the form $F(x_1, \ldots, x_n)$ with $|x_i| \leq B$.

For convenience, we employ a smooth non-negative weight $w : \mathbb{R}^n \to \mathbb{R}$ such that $w \geq 1$ on the unit cube $[-1/2, 1/2]^n$, is supported in $[-1, 1]^n$, and satisfies the differential inequality

$$\left|\frac{\partial^{\alpha}}{\partial \mathbf{x}^{\alpha}} w(\mathbf{x})\right| \ll_{\alpha} 1,$$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$. Define the normalized weight function

$$w_B(\mathbf{x}) = w(\mathbf{x}/B), \tag{5}$$

so that w_B is supported in the box $\mathcal{B} = [-B, B]^n$ and satisfies the differential inequalities

$$|\frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} w_B(\mathbf{x})| \ll_\alpha B^{-|\alpha|},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We then define the counting function

$$N_{w,B}(F) = \sum_{y \in \mathbb{Z}} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ F(\mathbf{x}) = y^r}} w_B(\mathbf{x}), \tag{6}$$

with the aim of proving upper bounds of the form

$$N_{w,B}(F) \ll B^{n-\delta},$$

for some $\delta > 0$ independent of the degree of F.

The counting function $N_B(\phi)$ associated to the cyclic cover $\phi : X \to \mathbb{P}^{n-1}$, where X is defined by (1), satisfies the relation

$$N_B(\phi) \leq N_{w,2B}(F).$$

Munshi employed the square sieve, adapted to count perfect r-th powers, in order to count solutions to (1). This method ultimately requires one to estimate mixed character sums, for which bounds Munshi employed bounds of Deligne [9] and Katz [21], [22]. We also use a version of the power sieve, but we sieve over certain almost-primes, instead of primes, and this allows us to apply the q-analogue of van der Corput's method; a similar combination has been used previously in [29], [19].

With no extra effort we can handle equations of the form (1) in which F is a general polynomial f, not necessarily homogeneous, whose leading form is nonsingular. Ultimately, we prove the following theorem for the counting function $N_{w,B}(f)$ defined as in (6) for solutions to the equation

$$y^r = f(x_1, \ldots, x_n). \tag{7}$$

Theorem A. *Let $f(\mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial of degree $d \geq 3$, and assume that its leading form is nonsingular. Then for any $r \geq 2$, the counting function $N_{w,B}(f)$ for the number of solutions to (7) satisfies*

$$N_{w,B}(f) \ll \begin{cases} B^{n-3n/(2n+10)}(\log B)^2, & n \geq 8, \\ B^{n-n(n-2)/(6n+4)}(\log B)^2, & 2 \leq n \leq 8, \end{cases}$$

where the implied constant depends on f, d, n.

The reader should recall that if f is a polynomial of degree d then its leading form is defined to be the form composed of all those terms in f with degree exactly d.

This theorem fails to hold for $d = 2$. The method of proof fails, but in fact the statement of the theorem is also false for $d = 2$ and $n > 10$, since it is well known that there are $\gg B^{n-1}$ values $\mathbf{x} \in [-B, B]^n$ for which $x_1^2 + \ldots + x_n^2$ is a square. (This follows from Theorems 5, 6 and 8 of Heath-Brown [16], for example.)

As an immediate corollary, we have:

Theorem B. *Let $\phi : X \to \mathbb{P}^{n-1}$ be a smooth finite cyclic cover given by the equation (1) with F a nonsingular form of degree mr. Suppose either that ϕ has degree $r \geq 3$, or that $r = 2$ and $m \geq 2$. Then*

$$N_B(\phi) \ll \begin{cases} B^{n-3n/(2n+10)}(\log B)^2, & n \geq 8, \\ B^{n-n(n-2)/(6n+4)}(\log B)^2, & 2 \leq n \leq 8. \end{cases}$$

In the case $r = 2$, $m = 1$, Theorem A no longer applies directly, but in this case the function F in (1) is a quadratic form, and it is well known that $N_{w,B}(F) \ll_F B^{n-1} \log B$. (See Theorems 5–8 of Heath-Brown [16], for example.) Assembling this with the relevant result of Theorem B for $r = 2$, we therefore have the following result for all smooth finite covers of degree 2 given by (1):

Theorem C. *Let $\phi : X \to \mathbb{P}^{n-1}$ be a smooth finite cover of degree 2, given by the equation (1) with F nonsingular. Then*

$$N_B(\phi) \ll \begin{cases} B^{n-1}(\log B)^2, & n \geq 10, \\ B^{9-27/28}(\log B)^2, & n = 9, \\ B^{n-n(n-2)/(6n+4)}(\log B)^2, & 2 \leq n \leq 8. \end{cases}$$

We therefore see that, for cyclic covers of any degree with F nonsingular, we can achieve Serre's conjecture (3) for $n \geq 10$, and indeed surpass it for $n > 10$ and degree $r \geq 3$. Moreover we improve on Munshi's bound (4) for $n \geq 8$.

4.1.C.2 Simultaneous prime values of pairs of quadratic forms

We describe here the completion of Objective A1, a two-dimensional Kloosterman refinement, with applications to simultaneous values of pairs of quadratic forms. This work will appear in the paper " Simultaneous prime values of pairs of quadratic forms," co-authored by Prof. D. R. Heath-Brown and Dr. L. B. Pierce. This manuscript is in the final stages of being prepared for submission to a peer-reviewed journal.

Given two quadratic forms $Q_1, Q_2 \in \mathbb{Z}[x_1, \ldots, x_k]$, we would like to understand which pairs of integers n_1, n_2 are represented simultaneously by Q_1 and Q_2. The situation for a single form is fairly well understood, but little is known for pairs of forms. Naturally there may be local obstructions to representability. However one might expect that if the variety V defined by the simultaneous equations

$$\begin{cases} Q_1(\mathbf{x}) &= 0 \\ Q_2(\mathbf{x}) &= 0 \end{cases} \tag{8}$$

is nonsingular, and if k is large enough, then every pair n_1, n_2 satisfying the necessary local conditions should be representable, or at least that all but finitely many such pairs are representable. In order to handle as small a value for k as possible, this paper asks only that "almost-all" suitable pairs are representable. By this we mean that the number of suitable pairs for which $|n_1|, |n_2| \leq N$ but n_1, n_2 are not simultaneously representable, should be $o(N^2)$ as N tends to infinity.

As a by-product of our investigation we will be able to say something about pairs of quadratic forms which simultaneously take prime values infinitely often. Certainly such pairs of forms exist: for example, Q_1 and Q_2 defined by

$$\begin{aligned} Q_1(x_1, x_2, x_3) &= x_1^2 + 4x_2^2 \\ Q_2(x_1, x_2, x_3) &= 4x_1^2 + x_3^2, \end{aligned}$$

simultaneously attain prime values infinitely often. Indeed one can use a result of Fouvry and Iwaniec to show that for "almost all" odd integers x_1 there is a prime of the form $x_1^2 + 4x_2^2$, and equally that for almost all such x_1 there is a prime of the form $4x_1^2 + x_3^2$. The claim then follows easily.

The geometric condition we must impose on the pair of quadratic forms is as follows:

Condition 1. *The projective variety defined over $\overline{\mathbb{Q}}$ by*

$$V : Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0 \tag{9}$$

is nonsingular; that is to say, for every $\mathbf{x} \in \overline{\mathbb{Q}}^k$*, if* $Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0$ *with* $\mathbf{x} \neq 0$ *then*

$$\mathrm{rk}\begin{pmatrix} \nabla Q_1(\mathbf{x}) \\ \nabla Q_2(\mathbf{x}) \end{pmatrix} = 2. \tag{10}$$

We will count solutions using a smooth non-negative weight $w : \mathbb{R}^k \to \mathbb{R}$ of compact support. We then define the normalized weight function

$$w_B(\mathbf{x}) = w(B^{-1}\mathbf{x}) \tag{11}$$

and the weighted representation function

$$R_B(n_1, n_2) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^k \\ \underline{Q}(\mathbf{x})=\underline{n}}} w_B(\mathbf{x}),$$

where we use an underscore to denote variables in 2-dimensional spaces, so that $\underline{Q}(\mathbf{x}) = (Q_1(\mathbf{x}), Q_2(\mathbf{x}))$, for example.

Our principal result is then the following, in which we set

$$F(x, y) = \det(xQ_1 + yQ_2).$$

Theorem D. *Suppose we have two quadratic forms* $Q_1, Q_2 \in \mathbb{Z}[x_1, \ldots, x_k]$ *satisfying Condition 1 with* $k \geq 5$*. Then if* $N = B^2$ *we have*

$$\sum_{\substack{\max\{|n_1|,|n_2|\}\leq N \\ F(n_2,-n_1)\neq 0}} |R_B(\underline{n}) - B\mathcal{J}(B^{-2}\underline{n}; w)\mathfrak{S}(\underline{n})^{k-4}|^2 \ll_{Q_1,Q_2,w} B^{2k-4-1/16}, \tag{12}$$

where $\mathcal{J}(\underline{\mu}; w)$ *and* $\mathfrak{S}(\underline{n})$ *are the singular integral and singular series (defined precisely in the paper).*

Of course this is of little use without some information about $\mathcal{J}(\underline{\mu}; w)$ and $\mathfrak{S}(\underline{n})$. This will be provided in our second result.

Theorem E. *Let* Q_1 *and* Q_2 *be as in Theorem D and suppose that* $k \geq 5$ *and* $F(n_2, -n_1) \neq 0$*. Then* $\mathfrak{S}(\underline{n}) \ll_{\varepsilon,Q_1,Q_2} \max(|n_1|, |n_2|)^{\varepsilon}$ *for any* $\varepsilon > 0$*. Moreover if the system of equations* $Q_1(\mathbf{x}) = n_1$*,* $Q_2(\mathbf{x}) = n_2$ *is solvable in every p-adic ring* $\mathbb{Z}_p$*, then* $\mathfrak{S}(\underline{n})$ *is real and positive. Indeed there is a constant* p_0 *depending only on* Q_1 *and* Q_2 *such that*

$$\mathfrak{S}(\underline{n}) \gg_{\varepsilon,Q_1,Q_2} \max(|n_1|, |n_2|)^{-\varepsilon} \prod_{p\leq p_0} |F(n_2, -n_1)|_p^{k-2},$$

for any fixed $\varepsilon > 0$*, where* $|\cdot|_p$ *denotes the standard p-adic valuation.*

Similarly, under the same conditions on $\underline{Q}$*, for any smooth* w *of compact support we have* $\mathcal{J}(\underline{\mu}; w) \ll_{Q_1,Q_2,w} 1$*. Moreover there is a constant* C*, depending on* Q_1 *and* Q_2*, with the following property. Suppose that* $w(\mathbf{x}) > 0$ *for* $|\mathbf{x}| \leq C$*. Then we will have*

$$\mathcal{J}(\underline{\mu}; w) \gg_{Q_1,Q_2,w} |F(\mu_2, -\mu_1)|^{k-2}$$

for any $\underline{\mu}$ *in the region* $1/2 \leq \max(|\mu_1|, |\mu_2|) \leq 1$ *for which the system of equations* $Q_1(\mathbf{x}) = \mu_1$*,* $Q_2(\mathbf{x}) = \mu_2$ *has a solution* $\mathbf{x} \in \mathbb{R}^k$*.*

As a corollary we have the following result.

Theorem F. *Suppose we have two quadratic forms $Q_1, Q_2 \in \mathbb{Z}[x_1, \ldots, x_k]$ that satisfy Condition 1 with $k \geq 5$. Let $\mathcal{E}(N)$ denote the number of integer pairs (n_1, n_2) with $|n_1|, |n_2| \leq N$ for which the system $Q_1(\mathbf{x}) = n_1$, $Q_2(\mathbf{x}) = n_2$ has a real solution and solutions in each $\mathbb{Z}_p$, but for which there is no solution $\mathbf{x} \in \mathbb{Z}^k$. Then*

$$\mathcal{E}(N) \ll_{Q_1, Q_2, \varpi} N^{2-\varpi},$$

with $\varpi = 1/(8k^3)$.

In particular, we may derive from Theorem F the following result on simultaneous prime values:

Theorem G. *Suppose two quadratic forms $Q_1(\mathbf{x}), Q_2(\mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_k]$ satisfy Condition 1 with $k \geq 5$. Suppose further that there is a real vector $\mathbf{x}_0$ such that $Q_1(\mathbf{x}_0), Q_2(\mathbf{x}_0) > 0$, and that for every prime q there is an integral vector $\mathbf{x}_q$ for which $q \nmid Q_1(\mathbf{x}_q)Q_2(\mathbf{x}_q)$. Then there are infinitely many pairs of primes simultaneously representable by $Q_1(\mathbf{x})$ and $Q_2(\mathbf{x})$.*

We prove that for diagonal quadratic forms, say $Q_1 = \sum a_i x_i^2$ and $Q_2 = \sum b_i x_i^2$, Condition 1 is equivalent to the condition that the ratios a_i/b_i are all distinct, for $i = 1, \ldots, k$. As a result, it is simple to find examples of pairs of forms that satisfy the requirements of Theorem G. For instance, one may take

$$\begin{aligned} Q_1(\mathbf{x}) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \\ Q_2(\mathbf{x}) &= x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 5x_5^2, \end{aligned}$$

for which the choices $\mathbf{x}_0 = (1, 0, \ldots, 0)$, $\mathbf{x}_q = (1, 0, \ldots, 0)$ clearly suffice.

We should remark at this point that one can handle 'target sets' other than the primes in much the same way. Provided that the target set has a counting function which grows faster than $N^{1-\varpi/2}$ all that is necessary is that one should be able to deal satisfactorily with the local conditions that arise. In particular we can handle the case in which the target set consists of the integers which are sums of two squares. This leads to an analytic proof of the following example of the Hasse Principle, (which is a special case of a far more general result due to Colliot-Thélène, Sansuc and Swinnerton-Dyer.

Corollary A. *Let $q_1(\mathbf{x}, \mathbf{y}) = Q_1(\mathbf{x}) - y_1^2 - y_2^2$, $q_2(\mathbf{x}, \mathbf{y}) = Q_2(\mathbf{x}) - y_3^2 - y_4^2$, where $Q_1, Q_2 \in \mathbb{Z}[x_1, \ldots, x_k]$ are quadratic forms satisfying Condition 1. Then the Hasse Principle holds for the intersection $q_1(\mathbf{x}, \mathbf{y}) = q_2(\mathbf{x}, \mathbf{y}) = 0$ as soon as $k + 4 \geq 9$.*

Here $k + 4$ is the total number of variables in the system $q_1(\mathbf{x}, \mathbf{y}) = q_2(\mathbf{x}, \mathbf{y}) = 0$. We leave the details of the proof to the reader.

We conclude the discussion of our results by observing that the range $k \geq 5$ is best possible in the sense that Theorem F would be false with $k = 4$. To justify this we first note that for any integer $L \geq 2$ the forms

$$Q_1(\mathbf{x}) = x_1^2 + x_3^2 + x_4^2, \quad Q_2(\mathbf{x}) = x_2^2 + x_3^2 + Lx_4^2$$

satisfy Condition 1. Moreover the equations

$$Q_1(\mathbf{x}) = n_1, \quad Q_2(\mathbf{x}) = n_2$$

will have a nonsingular solution in $\mathbb{F}_p$ for any $p \geq 11$ not dividing both L and n_2. Taking L as a large prime we conclude that the system has solutions over every $\mathbb{Z}_p$ providing that $n_1 \equiv n_2 \equiv 1 \pmod{840}$ and $L \nmid n_2$. Moreover there will be a real solution (with $x_3 = x_4 = 0$) whenever n_1 and n_2 are positive. Thus there are $\gg N^2$ pairs of positive integers $n_1, n_2 \leq N$ for which the local conditions are everywhere satisfied, with an implied constant independent of

L. However for a global solution one clearly has $|x_1|, |x_2|, |x_3| \leq N^{1/2}$ and $|x_4| \leq N^{1/2}L^{-1/2}$ so that there are $O(N^2 L^{-1/2})$ possible 4-tuples $(x_1, \ldots, x_4)$. Thus if L is chosen sufficiently large there will be $\gg N^2$ pairs n_1, n_2 for which there are local solutions but no global solution.

Our approach to proving these key theorems uses the two-dimensional circle method. The novel idea that allows us to reduce to $k \geq 5$ is a two-dimensional Kloosterman refinement applied to the contribution of the minor arcs.

The proofs of our results are long and involved, thus we now only sketch the method of proof schematically, as follows. With the notations introduced above we set

$$S(\alpha_1, \alpha_2) = \sum_{\mathbf{x} \in \mathbb{Z}^k} e(\alpha_1 Q_1(\mathbf{x}) + \alpha_2 Q_2(\mathbf{x})) w_B(\mathbf{x}), \tag{13}$$

for any $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, so that

$$R_B(n_1, n_2) = \int_0^1 \int_0^1 S(\alpha_1, \alpha_2) e(-\alpha_1 n_1 - \alpha_2 n_2) d\alpha_1 d\alpha_2.$$

With a suitable definition of the major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ in two dimensions, we may break R_B into the two pieces

$$R_B(n_1, n_2) = \iint_{\mathfrak{M}} + \iint_{\mathfrak{m}}$$

so that

$$\sum_{\underline{n} \in \mathbb{Z}^2} \left| R_B(n_1, n_2) - \iint_{\mathfrak{M}} S(\alpha_1, \alpha_2) e(-\alpha_1 n_1 - \alpha_2 n_2) d\alpha_1 d\alpha_2 \right|^2$$
$$= \sum_{\underline{n} \in \mathbb{Z}^2} \left| \iint_{\mathfrak{m}} S(\alpha_1, \alpha_2) e(-\alpha_1 n_1 - \alpha_2 n_2) d\alpha_1 d\alpha_2 \right|^2. \tag{14}$$

Temporarily set $f(\alpha_1, \alpha_2) = S(\alpha_1, \alpha_2) \chi_{\mathfrak{m}}(\alpha_1, \alpha_2)$. Then the right hand side of (14) is

$$\begin{aligned} \sum_{\underline{n} \in \mathbb{Z}^2} \left| \iint_{[0,1]^2} f(\alpha_1, \alpha_2) e(-\alpha_1 n_1 - \alpha_2 n_2) d\alpha_1 d\alpha_2 \right|^2 &= \iint_{[0,1]^2} |f(\alpha_1, \alpha_2)|^2 d\alpha_1 d\alpha_2 \\ &= \iint_{\mathfrak{m}} |S(\alpha_1, \alpha_2)|^2 d\alpha_1 d\alpha_2, \end{aligned} \tag{15}$$

where in the first equality we have applied Parseval's identity.

We will show that if $\max\{|n_1|, |n_2|\} \leq N = O(B^2)$ and $F(n_2, -n_1) \neq 0$ the contribution of the major arcs may be approximated as

$$\iint_{\mathfrak{M}} S(\alpha_1, \alpha_2) e(-\alpha_1 n_1 - \alpha_2 n_2) d\alpha_1 d\alpha_2 = M(n_1, n_2) + E(n_1, n_2), \tag{16}$$

where $M(n_1, n_2)$ is the expected main term, including the singular series and the singular integral, and $E(n_1, n_2)$ is an acceptable error term. Thus it will follow that

$$\sum_{\substack{\max\{|n_1|, |n_2|\} \leq N \\ F(n_2, -n_1) \neq 0}} |R_B(n_1, n_2) - M(n_1, n_2)|^2 \ll \iint_{\mathfrak{m}} |S(\alpha_1, \alpha_2)|^2 d\alpha_1 d\alpha_2 + E(N), \tag{17}$$

where

$$E(N) = \sum_{\underline{n} \in \mathbb{Z}^2} |E(n_1, n_2)|^2.$$

When $\max(|n_1|, |n_2|)$ is of order B^2 the expected size of $M(n_1, n_2)$ is B^{k-4}, since roughly speaking, this is the probability that as $\mathbf{x}$ ranges over B^k possible choices, both of the values $Q_1(\mathbf{x}) - n_1$ and $Q_2(\mathbf{x}) - n_2$ (each of size up to $O(B^2)$), is zero. Imagine for the moment that $M(n_1, n_2) \gg B^{k-4}$; this implicitly assumes that the singular series and singular integral have suitable lower bounds. (In fact, proving such lower bounds is a significant source of complication.) Suppose furthermore that the right hand side of (17) is bounded by B^β for some $\beta > 0$. Under these two significant assumptions, (17) would imply that

$$\#\{(n_1, n_2) \in \mathbb{Z}^2 : \max(|n_1|, |n_2|) \ll B^2 : R_B(n_1, n_2) = 0\} \cdot (B^{k-4})^2 \ll B^\beta, \tag{18}$$

whence

$$\#\{(n_1, n_2) \in \mathbb{Z}^2 : \max(|n_1|, |n_2|) \le N : R_B(n_1, n_2) = 0\} \le B^{\beta-(2k-8)}$$

for $N \ll B^2$. If β is such that the right hand side is $o(N^2)$, then we may conclude that almost all pairs of integers n_1, n_2 of size N are represented simultaneously by Q_1, Q_2.

Thus we seek to bound (17) by B^β with $\beta < 2k-4$. The mean square argument above has appeared before in settings in which the circle method is used to show that almost all integers (in some suitable sense) are represented by a particular form. It dates from the work of Hardy and Littlewood [13] who showed, for example, that almost-all positive integers are the sum of 5 non-negative cubes.

Our main goals now are therefore a representation of the contribution of the major arcs in the form (16), and a bound for the right hand side of (17) by a precise upper bound of the form $o(B^{2k-4})$. In particular, this will require a bound for the integral over the minor arcs of the shape

$$\iint_{\mathfrak{m}} |S(\alpha_1, \alpha_2)|^2 d\alpha_1 d\alpha_2 = o(B^{2k-4}). \tag{19}$$

It is here that the main innovation of our work lies. Note that the application of Parseval's identity in (15) has achieved several things: first, it has raised the weighted exponential sum $S(\alpha_1, \alpha_2)$ to a higher power and homogenized the problem, thus passing the problem of counting simultaneous solutions of $Q_1(\mathbf{x}) = n_1$, $Q_2(\mathbf{x}) = n_2$ to the problem of counting simultaneous solutions of

$$Q_1(\mathbf{x}_1) - Q_1(\mathbf{x}_2) = 0, \quad Q_2(\mathbf{x}_1) - Q_2(\mathbf{x}_2) = 0,$$

where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Z}^k$. Moreover, it has passed the power of $S(\alpha_1, \alpha_2)$ inside the relevant integral, thus increasing the possibility for averaging advantageously. In order to do so, we must treat the double integral over α_1, α_2, and the possible interactions between α_1 and α_2, nontrivially. We accomplish this by developing a two-dimensional Kloosterman refinement.

Historically, the Kloosterman refinement applies to a generating function of a single variable, say

$$F(\alpha) = \sum_n r(n) e(\alpha n),$$

in which case one is interested in computing the representation number

$$r(0) = \int_0^1 F(\alpha) d\alpha.$$

To do so, one would traditionally apply the circle method to estimate portions of the integral with $\alpha \approx a/q$ for certain rational numbers a/q. Kloosterman's innovation [23] was to exploit cancellation between the contributions corresponding to pairs of distinct rationals $a_1/q, a_2/q$ with a common denominator q.

As suggested in [16], it would be desirable to apply a Kloosterman refinement to two-dimensional problems, in which case one would consider

$$\iint_{[0,1]^2} \sum_{m,n} r(m,n) e(\alpha_1 m + \alpha_2 n) d\alpha_1 d\alpha_2,$$

and hope to extract cancellation between portions of the integral corresponding to neighbourhoods of distinct pairs of rationals $(a_1/q, a_2/q)$ and $(b_1/q, b_2/q)$. To do so, one would need to approximate α_1, α_2 simultaneously by rationals with the same denominator, so that the intervals in the refinement all share the same length. This can be accomplished by a 2-dimensional application of Dirichlet's principle, which guarantees for any $S \geq 1$ the existence of $1 \leq a_1, a_2 \leq q \leq S$, with $(a_1, a_2, q) = 1$ such that

$$|\alpha_1 - a_1/q| \leq \frac{1}{q\sqrt{S}}, \qquad |\alpha_2 - a_2/q| \leq \frac{1}{q\sqrt{S}}. \tag{20}$$

In the corresponding case in one dimension, Dirichlet's approximation principle places α in an interval of length $(qS)^{-1}$, and one can show that α can lie in at most two such intervals, if $q \leq S$. In two dimensions, the intervals are longer, and a given pair (α_1, α_2) may lie in many such intervals simultaneously. This makes a true 2-dimensional Kloosterman refinement difficult to carry out.

The key feature in our application is that we apply the 2-dimensional Kloosterman refinement only to the minor arcs contribution, for which we need simply an upper bound rather than an asymptotic. Thus we include all approximations (20) (accepting the possible overlap and resulting loss in sharpness) and then carry out a procedure to extract cancellation between the contributions of distinct pairs of rationals. We note that while in a 1-dimensional application of the Kloosterman refinement one typically encounters exponential sums involving both a and its inverse $\overline{a}$ modulo q, in our particular 2-dimensional application, such inverses do not appear. This is because in the usual 1-dimensional version one has to take detailed account of the end points of the Farey arcs, while in our 2-dimensional situation it suffices to use the simple explicit squares (20).

It is worth remarking that if one simply wanted to prove a result such as Theorem F for any $k > k_0$ sufficiently large one could avoid the Kloosterman refinement. In particular, one could use the simpler methods of Birch [1] for the treatment of the major arcs for k sufficiently large. But in order to push the number of variables down to 5, we must use the more technical argument presented in this paper.

4.1.C.3 Carleson operators of Radon type

One of Dr. Pierce's main interests is the field of arithmetic operators. These are discrete analogues of Radon transforms (which classically are defined over Euclidean space), and for the most part their behavior remains mysterious. Understanding these operators requires in many cases a deep understanding of related questions in number theory (many of which remain open currently). Dr. Pierce is interested in developing techniques such as the circle method (as in Objective A) in order to prove new results for arithmetic operators. Typically, an application of the circle method splits the problem of bounding the discrete operator in question into two parts: first, bounding a character sum, and second, bounding a classical operator on functions of Euclidean space. Thus while understanding the original operator in the continuous setting is not usually sufficient for bounding its discrete analogue, it may be necessary.

Inspired by her work on the circle method and quadratic forms (Objective A1), Dr. Pierce developed an interest in Carleson operators of Radon type. This initiated a major long-term project in collaboration with Dr. P. L. Yung, which we summarize here. This project has resulted in a significant result, which will appear in the paper "Polynomial Carleson Operators of Radon Type," co-authored by Dr. L. B. Pierce and Dr. P. L. Yung. This manuscript is in the final stages of being prepared for submission to a peer-reviewed journal. A second phase of this project will address the discrete analogue, and will require the application of circle method techniques, as related to Objective A.

A celebrated theorem of Carleson [7] proves an L^2 bound for the operator

$$f \to \sup_{\lambda \in \mathbb{R}} |T_\lambda f(x)|, \tag{21}$$

where for each $\lambda \in \mathbb{R}$,

$$T_\lambda f(x) = p.v. \int_{-\infty}^{\infty} f(x-y) e^{i\lambda y} \frac{dy}{y}.$$

This result answered in the affirmative a long-standing question of Luzin on whether the Fourier series of an L^2 function must converge pointwise almost everywhere. Carleson's remarkable result, and his method, which involved time-frequency analysis, inspired many generalizations. Hunt proved L^p bounds for $1 < p < \infty$ for the Carleson operator (21), and Sjölin [34] introduced the Carleson operator in the setting of $\mathbb{R}^n$, by defining

$$T_\lambda f(x) = \int_{\mathbb{R}^n} f(x-y) e^{i\lambda \cdot y} K(y) dy,$$

for an appropriate class of Calderón-Zygmund kernels K (to be defined later). He proved that

$$f \to \sup_{\lambda \in \mathbb{R}} |T_\lambda f(x)|,$$

is a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Further significant generalizations and applications of Carleson's method were developed by Fefferman [10] and Lacey and Thiele [24].

One could naturally ask if L^p bounds continue to hold when the linear phase $\lambda \cdot y$ is replaced by a real polynomial

$$P_\lambda(y) = \sum_{1 \leq |\alpha| \leq d} \lambda_\alpha y^\alpha \tag{22}$$

of fixed degree d. One then defines

$$T_\lambda f(x) = \int_{\mathbb{R}^n} f(x-y) e^{iP_\lambda(y)} K(y) dy, \tag{23}$$

where $K(y)$ is again chosen to be an appropriate Calderón-Zygmund kernel. The goal is then to show that

$$\| \sup_\lambda |T_\lambda f(x)| \|_{L^p(\mathbb{R}^n)} \leq A \|f\|_{L^p(\mathbb{R}^n)}, \tag{24}$$

for all $1 < p < \infty$, where the supremum is now taken over all sets of coefficients $\lambda = (\lambda_\alpha)_\alpha$ with each λ_α ranging over $\mathbb{R}$. This operator, whose study was initiated by Elias M. Stein, is now known as the polynomial Carleson operator, and remains mysterious in many cases.

A convenient way to formulate the supremum is to define a stopping-time function $\lambda(x)$, which is taken to be any measurable function mapping $\mathbb{R}^n$ to the space of real coefficients for $P_\lambda(y)$. Then the desired bound (24) would be implied by a bound of the form

$$\| |T_{\lambda(x)} f(x)| \|_{L^p(\mathbb{R}^n)} \leq A \|f\|_{L^p(\mathbb{R}^n)},$$

with norm independent of the choice of the stopping-time function $\lambda(x)$.

Along these lines, it was first shown by Ricci and Stein [32] that for any fixed polynomial $P(x, y)$, the operator

$$Tf(x) = \int_{\mathbb{R}^n} f(x-y) e^{iP(x,y)} K(y) dy$$

is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, with norm dependent only on the degree of $P(x, y)$ and not on the coefficients; this is the case of a polynomial stopping-time function, and does not imply the full result of (24). In another direction, Stein [36] showed that in the specific case of $\mathbb{R}^1$ with purely quadratic phase polynomial $P_\lambda(y) = \lambda y^2$, one could prove that for

$$T_\lambda f(x) = \int_{\mathbb{R}} f(x-y) e^{iP_\lambda(y)} K(y) dy, \tag{25}$$

the corresponding maximal operator

$$f \mapsto \sup_{\lambda \in \mathbb{R}} |T_\lambda f(x)|$$

is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. The methods used in this case hinged upon an asymptotic for the Fourier transform of the kernel $e^{i\lambda y^2}/y$, which is not easily generalizable to higher powers in the phase, or to higher dimensions.

In 2001, Stein and Wainger [38] introduced a new approach to polynomial Carleson operators, based on Van der Corput estimates for oscillatory integrals, as well as a Kolmogorov-Seliverstov stopping-time argument. Their result is as follows:

Theorem H. *The operator T_λ defined in (23) satisfies*

$$\| \sup_\lambda |T_\lambda f(x)| \|_{L^p(\mathbb{R}^n)} \leq A \|f\|_{L^p(\mathbb{R}^n)},$$

for $1 < p < \infty$, as long as $P_\lambda(y)$ omits linear terms.

The restriction that $P_\lambda(y)$ omits linear terms is inherent to methods of Van der Corput type, as will be seen explicitly later.

Until very recently, bounds for the full polynomial Carleson operator remained unproved. Now, the case of the polynomial Carleson operator in dimension one has been resolved by Victor Lie , who proves by time-frequency analysis methods that for

$$C_\lambda f(x) = p.v. \int_{\mathbb{T}} f(x-y) e^{iP_\lambda(y)} \frac{dy}{y},$$

where $\mathbb{T} = [-1/2, 1/2]$, the corresponding maximal version extends to a bounded operator on L^p for $1 < p < \infty$ with

$$\| \sup_\lambda |C_\lambda f| \|_{L^p(\mathbb{T})} \leq A \|f\|_{L^p(\mathbb{T})}.$$

The case of the polynomial Carleson operator involving both linear and higher order terms in dimensions $n \geq 2$ remains open.

In this paper, we take the polynomial Carleson operator in an entirely new direction, by introducing Radon type behavior. Precisely, for any polynomial P_λ as defined in (22), we define an operator, initially acting on functions f of Schwartz class, by

$$T_\lambda f(x,t) = \int_{\mathbb{R}^n} f(x-y, t-|y|^2) e^{iP_\lambda(y)} K(y) dy. \tag{26}$$

Thus T_λ integrates f along the paraboloid $(y, |y|^2) \subset \mathbb{R}^{n+1}$ against an oscillatory factor and a singular kernel. The kernel K is a Calderón-Zygmund kernel, that is a tempered distribution agreeing with a C^1 function $K(x)$ for $x \neq 0$, that it satisfies the differential inequalities

$$|\partial_x^\alpha K(x)| \leq A|x|^{-n+|\alpha|} \quad \text{for } 0 \leq |\alpha| \leq 1, \tag{27}$$

and that $\hat{K}$ is an L^∞ function.

We then consider the polynomial Carleson operator of Radon type defined by

$$Tf(x,t) = \sup_\lambda |T_\lambda f(x,t)|$$

where the supremum is taken over all coefficients of the phase $P_\lambda(y)$. It is natural to expect that T extends to a bounded operator on $L^p(\mathbb{R}^{n+1})$ for all $1 < p < \infty$, yet a result of this type seems out of reach via time-frequency methods, at present.

Therefore we develop an approach for treating T with Van der Corput estimates, inspired by the work of Stein and Wainger in [38]. Such methods inherently require that the phase polynomial lack terms of certain lower orders; in the case of the paraboloid, the restriction that arises is that the polynomial phase $P_\lambda(y)$ lacks both linear and quadratic terms. Thus we would expect for Van der Corput methods to be able to prove that T is a bounded operator on $L^p(\mathbb{R}^{n+1})$ as long as the phase polynomial $P_\lambda(y)$ belongs to the class $\mathcal{P}_{[3,d]}$ of polynomials of degree at most d that vanish to at least order 3 at the origin, i.e. of the form

$$P_\lambda(y) = \sum_{3\leq|\alpha|\leq d} \lambda_\alpha y^\alpha.$$

However, our main result for T currently holds for polynomials belonging to the following more restrictive class:

Definition. We define the class $\mathcal{S}_{[3,d]}$ of *allowable sparse polynomials* to be polynomials $P_\lambda(y)$ of the form

$$P_\lambda(y) = \sum_{1\leq l\leq d} \lambda_l \sum_{\alpha\in\Upsilon(l)} c_\alpha y^\alpha,$$

where c_α are *fixed* nonzero real coefficients, and

1. for each $1 \leq l \leq d$, $\Upsilon(l)$ is a fixed subset of the set of all multi-indices of order l
2. $\Upsilon(1) = \Upsilon(2) = \emptyset$
3. if for $l_1 \neq l_2$, both $\Upsilon(l_1)$ and $\Upsilon(l_2)$ are nonempty, then $|l_1 - l_2| \geq 2$.
4. for every even degree l such that $\Upsilon(l) \neq \emptyset$, there exists an index $\alpha = (\alpha_i)_i$ with α_i even for all i such that $\alpha \notin \Upsilon(l)$.

An example of such a polynomial is $P_\lambda(y) = \lambda_3 y_1^3 + \lambda_5 y_1^3 y_2^2$.

We may now state our theorem for the polynomial Carleson operator of Radon type:

Theorem I. *For any $n \geq 2$, let T_λ be defined as in (26) with polynomial phase $P_\lambda(y)$ belonging to the class $\mathcal{S}_{[3,d]}$. Then for every $1 < p < \infty$, we have an* a priori *inequality for functions f in Schwartz class:*

$$\| \sup_\lambda |T_\lambda f| \|_{L^p(\mathbb{R}^{n+1})} \leq A_p \|f\|_{L^p(\mathbb{R}^{n+1})}, \tag{28}$$

where the supremum runs over $\lambda = (\lambda_\alpha)_\alpha$ coefficients for $P_\lambda(y)$.

As a result of the *a priori* inequality, it follows from conventional limiting arguments that the Carleson operator extends to a bounded operator on L^p itself, for which the same bound (28) holds.

We make here a few remarks on the method of proof, which was initially inspired by the work of Stein and Wainger in [38]. In the broadest terms, our method of proof is to split for any set of coefficients λ, the integral defining T_λ in two parts, one in which y is small enough, relative to the coefficients λ, that $e^{iP_\lambda(y)}$ does not contribute much oscillation; this part we compare to maximal truncated singular Radon transforms, which we may bound on L^p with conventional methods. The second part comprises the region in which y is large enough, relative to λ, that $P_\lambda(y)$ exhibits significant oscillation; this part we reduce to a crucial L^2 bound for an auxiliary maximal oscillatory Radon transform, which is interesting in its own right.

Define

$$^{(\eta_a)}I_a^\lambda f(x,t)f(x,t) = \int_{\mathbb{R}^n} f(x-y, t-|y|^2)e^{iP_\lambda(y/a)}\frac{1}{a^n}\eta_a(\frac{y}{a})dy. \tag{29}$$

Here $a > 0$ is a real number, and η_a belongs to a one-parameter family of bump functions that are supported in the unit ball and are uniformly in C^1; that is, there exists a constant Λ such that for all $a > 0$,

$$\|\eta_a\|_{C^1} := \sup_{y\in B_1} |\eta(y)| + \sup_{y\in B_1} |\nabla\eta(y)| \leq \Lambda.$$

Also, $\lambda = (\lambda_\alpha)_\alpha$ denotes the coefficient parameter, and we may quantify the size of the coefficient parameter λ by defining the norm $\|\lambda\| = \sum_{1\leq|\alpha|\leq d}|\lambda_\alpha|$.

Our principal result for this operator is as follows:

Theorem J. *For $n \geq 2$, suppose that $P_\lambda(y)$ is a polynomial belonging to one of the classes allowed in Theorem I. There exists some fixed $\delta > 0$ such that for every $r \geq 1$,*

$$\| \sup_{\substack{\|\lambda\|\geq r \\ k\in\mathbb{Z}}} |^{(\eta_{2^k})}I_{2^k}^\lambda f(x,t)| \|_{L^2(\mathbb{R}^{n+1})} \leq A\Lambda r^{-\delta}\|f\|_{L^2(\mathbb{R}^{n+1})}. \tag{30}$$

The proof of Theorem J is the most difficult and novel aspect of the paper. Stein and Wainger's original approach was to apply a TT^* argument to an operator analogous to $^{(\eta_a)}I_a^\lambda$, and then apply Van der Corput bounds to the kernel of TT^*, in order to show that TT^* could be majorized by certain maximal functions for which L^2 bounds are known. In our setting of Radon transforms, this approach is not sufficient.

Instead, we were inspired by the treatment of the maximal Radon transform on the paraboloid (see [37] and §1.2 of Chapter 11 in [35]) to introduce a smoother version of the operator I_a^λ, as well as an accompanying square function. Proving that the square function is bounded on L^2 is the heart of proving Theorem J.

We further remark on the restrictions in Theorems J and hence Theorem I. The restriction that $P_\lambda(y)$ omit linear and quadratic terms arises from the application of van der Corput methods. Consider an operator

$$Rf(x,t) = \int_{\mathbb{R}^n} f(x-y,t-|y|^2)\eta(y)e^{iP_\lambda(y)}dy = \int_{\mathbb{R}^n} f(x-y,t-z)\eta(y)e^{iP_\lambda(y)}\delta_{z=|y|^2}dydz, \tag{31}$$

where η is a C^1 bump function of compact support. If λ is regarded as fixed for the moment, we may compute the Fourier multiplier of R, namely

$$m(\xi,\theta) = \int_{\mathbb{R}^n} \eta(y)e^{iP_\lambda(y)-2\pi i\xi\cdot y-2\pi i\theta|y|^2}dy. \tag{32}$$

We could obtain a good upper bound for m_2 by applying van der Corput bounds, as long as we had good lower bounds for the coefficients of the polynomial phase $P_\lambda(y) - 2\pi\xi\cdot y - 2\pi\theta|y|^2$. However, intuitively, we could not expect to obtain such lower bounds for the coefficients (especially for the linear or quadratic coefficients) if λ were an unknown stopping-time function. Of course, if $\lambda = \lambda(x,t)$ is a stopping-time function, we could not compute a Fourier multiplier, so this is merely a heuristic, but in fact we see an explicit demonstration of this phenomenon in our proof.

4.1.C.5 Ongoing and future collaborations

One of the key goals of this project was to solidify a long-term collaboration between Prof. D. R. Heath-Brown and Dr. L. B. Pierce, with the intention that work on certain aspects of the project would continue past the conclusion of the funding period. This has been accomplished, and the collaboration continues on several problems (in particular aspects of Objective D).

A second key goal of the project was to connect Dr. Pierce with the international research scene and to facilitate her establishment as a researcher and collaborator. This has also been accomplished; we describe briefly below two further ongoing research interests that she has initiated with mathematicians in the international community (in particular related to Objective E). (As stated in the original proposal, Objective E was not expected to be completed during the two years of funding, but was anticipated to form a basis for future collaboration, as it is now doing.)

Note on Objectives A2 and A3: Ongoing work of Prof. C. Hooley on cubic forms motivated us to refrain from engaging on research on Objective A2. As expected in the timeline drafted in the initial proposal, there was not sufficient time to pursue the approach proposed for Objective A3 during the two years. Future work of Dr. L. B. Pierce will address the question from the standpoint of ℓ^p boundedness properties of discrete fractional integral operators.

Note on Objective B: exciting new work by Prof. T. Wooley has redrawn the landscape for mean values of Weyl sums. While this deep new work is unfolding, it was thought best to set aside temporarily the related work of Objective B, which would possibly prove redundant relative to Wooley's ongoing programme. We will return to this idea in the near future if it again appears relevant.

4.1.C.5a Burgess bounds

This describes ongoing work of Prof. Heath-Brown and Dr. Pierce, relevant to Objective D. Many people have tried to generalise Burgess' results to polynomials by considering sums of the form $\sum_n \chi(f(n))$. Such attempts have had very limited success, and as a result it has appeared unwise to contemplate polynomials in more than one variable, given the failures on polynomials in one variable. However, the viewpoint of this proposal is different. Instead we view Burgess' bound as relating to sums of the form $\sum_{n\in I} \chi(n)$, where I is an arbitrary interval. If one replaces $\chi(n)$ by $\chi(f(n_1,...,n_k))$, where f is homogeneous, and $n_1,...,n_k$ run over a box, we anticipate that the major structural difficulties encountered with general polynomials in one variable will not appear.

Thus, given a homogeneous form $f(\mathbf{x})$ of degree d in n variables, define for any prime p the Legendre character sum

$$S_p(N) = \sum_{|x_j|\leq N} \left(\frac{f(\mathbf{x})}{p}\right).$$

Deligne's results give the bound $S_p(N) = O(p^{n/2+\epsilon})$ for suitable forms f. When f is linear in one variable, and for any nontrivial Dirichlet character in place of the Legendre symbol, Burgess [5] provides nontrivial bounds. It appears to be possible to extend Burgess' original method to general forms of degree d, and Prof. Heath-Brown and Dr. Pierce are investigating this possibility fully.

4.1.C.5b Nonlinear Schrödinger equation in the periodic case

Bourgain [2] has proposed the study of the NLS in the periodic case, namely solutions u of

$$i\partial_t u + \Delta u - u|u|^{p-2} = 0,$$

with $u(x,t) \in H^1(\mathbb{T}^d)$ for $t = 0$, and dimension $d \leq 3$ (and assuming $p < 6$ if $d = 3$). Bourgain's work partially resolves the question of Strichartz estimates if $M = \mathbb{T}^d$ is the usual torus, that is with the Laplacian taking the form

$$(e^{it\Delta}\phi)(x) = \sum_{n\in\mathbb{Z}^d} \hat{\phi}(n)e^{2\pi i(nx+|n|^2t)}.$$

More questions remain open in the case where M is an irrational torus, with

$$(e^{it\Delta}\phi)(x) = \sum_{n\in\mathbb{Z}^d} \hat{\phi}(n)e^{2\pi i(nx+Q(n)t)},$$

where

$$Q(n) = \theta_1 n_1^2 + \cdots + \theta_d n_d^2$$

and the θ_i are irrational. Bourgain has isolated certain questions of number theoretic interest that control the Strichartz estimates in the case of the irrational torus, and Pierce is currently investigating ways to improve the existing estimates using number theory (Objective E).

4.1.C.5c Second moments of the 3-part of class numbers

This describes ongoing work of Dr. Pierce and Dr. Frank Thorne, related to Objectives A, C, and D. For a square-free integer (or fundamental discriminant) d, consider the quadratic field $\mathbb{Q}(\sqrt{d})$ with class group $CL(d)$ and class number $h(d)$. The 3-part $h_3(d)$ is defined by

$$h_3(d) = \#\{[\mathfrak{a}] \in CL(d) : [\mathfrak{a}]^3 = 1\}.$$

This admits the trivial bound

$$h_3(d) \leq h(d) \ll |d|^{1/2+\varepsilon}. \tag{33}$$

Conjecturally,

$$h_3(d) \ll |d|^{\epsilon} \tag{34}$$

for any $\epsilon > 0$. In the case of real fields, it is conjectured that for most $d > 0$, $h(d)$ is itself very small. In the case of imaginary fields, when $d < 0$, the conjecture (34) is more significant, as it indicates that although $h(d)$ itself tends to infinity at least as fast as $|d|^{1/2-\varepsilon}$ for any $\varepsilon > 0$, the 3-part is conjectured to be very small. (Note that a known bound for $h_3(d)$ for $d < 0$ implies a bound of the same order of magnitude for $h_3(d')$ for related $d' > 0$, by the reflection principle of Scholz.)

Dr. Pierce [27], [28], [29] obtained several nontrivial bounds for $h_3(d)$ (this relates to the techniques of Objective C). The current best known bound is

$$h_3(d) \ll |d|^{1/3+\varepsilon},$$

due to Ellenberg and Venkatesh. This is still far from the conjectured upper bound, and currently people are interested not only in pointwise upper bounds but also upper bounds on average.

Davenport and Heilbronn did groundbreaking work on averages of the 3-part, showing (very roughly speaking) that

$$\sum_{0<d<X} h_3(d) \approx X, \qquad \sum_{-X<d<0} h_3(d) \approx X.$$

This is in line with the expectation (34). These results have recently been refined, to include lower order terms.

Dr. Pierce's current goal, in collaboration with Dr. Frank Thorne, is to prove bounds toward the expected behavior of the second moment, namely

$$\sum_{0<d<X} (h_3(d))^2 \ll X, \qquad \sum_{-X<d<0} (h_3(d))^2 \ll X.$$

This project ties in closely to the themes of the Marie Curie Objectives A, C, and D, because bounds for exponential sums, the circle method, and an understanding of values of homogeneous forms all play a role in our approach to the problem.

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4.1.D Potential impact and main dissemination activities

4.1.D.1 Impact on the research landscape

The work of this long-term research programme has the potential to influence current mathematical research ranging from analytic number theory to harmonic analysis. The work on power sieves has developed a new tool with wide applicability to problems in number theory, and has pushed forward current understanding of a conjecture of Serre. The work on the circle method has broken a long-standing barrier and pushed the circle method to its theoretical limit within the context of the problem at hand; this has also opened the door to new and advantageous applications of the circle method to higher dimensional problems. The work on Carleson operators of Radon type has introduced an entirely new element to the setting of Carleson operators, and it is expected that this will generate wide-ranging interest in such problems.

4.1.D.2 Creating long term collaborations and mutually beneficial co-operation between Europe and the US

This project has solidified a long term collaboration between Lillian Pierce and Prof. Heath-Brown in analytic number theory. The research projects have opened up new questions and methods, which will provide future projects for the collaboration. In particular, the core problems, Objectives A1, C and D have opened up many new interesting avenues for research, which the researchers will investigate in continuing work. In addition, the optional problems (such as Objective E) are providing material for a smooth segue into collaboration with further international researchers at the end of the two years. Furthermore, the problems described in this proposal are of great interest to number theorists both in Europe and in North America, and the collaboration between Pierce and Prof. Heath-Brown has produced results that have been disseminated widely on both continents.

Pierce is now strongly placed to maintain productive collaborations with researchers in both Europe and the US. At the completion of the Marie Curie funding period, Pierce is now a Research Fellow at the Mathematical Institute at Oxford, funded by the National Science Foundation of the USA. In 2013-2014, Pierce will be a Bonn Junior Fellow (w-2 professor) at the Hausdorff Centre for Mathematics in Bonn, Germany. In 2014, she will join the faculty at Duke University in the US. The opportunity for her to receive advanced training at Oxford and to initiate collaborations with mathematicians in Europe has now established Pierce's strong long-term links between research institutions in Europe and the US.

4.1.D.3 Contribution to European excellence and European competitiveness

This project has contributed to European excellence and competitiveness. First, Pierce brought her first-rate educational background to the number theory group, contributing a wide range of skills and an enthusiasm for sharing knowledge to the group. Second, the proposed research, having taken place at Oxford, and being disseminated widely in Europe and in North America, has advanced the reputation of European number theory, making Europe even more attractive to researchers in the future. Third, the proposed work paved the way for future collaborations between Pierce and researchers in Europe, which will not only stimulate number theory research in Europe but will also serve to broadcast the quality of research in Europe. Fourth, having received training in Europe, Pierce is now in a position to recommend similar experiences to colleagues and students in the future, not only strengthening the links between research in Europe and the US, but publicising the quality of European programmes, thus potentially stimulating a long-term influx of international researchers to mathematics centres in Europe. In particular, Pierce has been awarded an internationally competitive Bonn Junior Fellow position at the Hausdorff Centre for Mathematics in Bonn.

4.1.D.4 Contribution to career development

Pierce gained specific mathematical knowledge and research experience by working on the proposed problems at Oxford under the mentorship of Prof. Heath-Brown. He is an expert on the material surrounding the proposed problems and played a key role in helping Pierce gain a foundational knowledge of all the relevant tools necessary to solve the problems. Under his tutelage, Pierce learned about his approach to a broad range of problems in analytic number theory, and gained a valuable overview of current trends in the field. Second, Pierce gained general knowledge in number theory and related fields by attending and presenting in ongoing number theory seminars at Oxford. Third, Pierce had frequent interactions with the lively number theory group at Oxford, providing frequent opportunities for stimulating mathematical conversations, as well as more casual opportunities to ask for guidance on research. She also organized a retreat for the number theory group, called the Oxford Number Theory Day (October 2012). Pierce gained valuable experience in lecturing, both on classical mathematics and on her own research, through the many opportunities available at Oxford and in the UK and Europe for giving lectures and research seminars. She presented her research at seminars in mathematics departments throughout Europe and the US, and learned valuable mentoring skills by interacting with senior professors, such as Prof. Heath-Brown, who have proven expertise as educators and mentors.

The Marie Curie fellowship has clearly advanced Pierce's career; at its conclusion, she has been a sought-after candidate for tenure-track jobs both in the US and abroad. Pierce was invited to interview for over 30 tenure-track faculty positions in the US. She has chosen to accept a Bonn Junior Fellow faculty position at the Hausdorff Centre for Mathematics (Bonn, Germany), after which she will join the faculty at Duke University in the United States.

4.1.D.5 Dissemination activities

The results of the funding time-period have been disseminated in peer-reviewed papers, in an open access online repository (ArXiv), on the project webpage, and through presentations of Dr. Pierce at research seminars in the US and Europe as well as international conferences. As a result of the success of this project, Dr. Pierce has been an invited speaker and participant for the following events (listed in reverse chronological order):

Upcoming invitations in 2013
Oct Women in Numbers - Europe, Luminy, France
Oct Oberwolfach Workshop on Analytic Number Theory, Germany
Jun Romanian Mathematical Society meeting, Alba Iulia, Romania
Jun Analytic Number Theory, Étienne Fouvry 60th birthday conference, Luminy, France
May Collaborative Analytic Number Theory Workshop, University of Illinois at Chicago
May University of Helsinki Colloquium, Finland
Apr Danish Mathematical Society 140th Anniversary Celebration, Denmark

Seminars and Conferences in 2013
Jan Caltech Colloquium
Jan Penn State Special Lecture and Number Theory Seminar
Jan University of Illinois at Urbana-Champaign Colloquium
Jan University of Minnesota Special Lecture
Jan MIT Analysis Seminar and Number Theory Seminar
Jan McGill University Special Lecture
Jan University of Wisconsin, Madison Colloquium
Jan Number Theory Seminar, AWM Workshop, Joint Mathematics Meetings, San Diego

Seminars and Conferences in 2012
Dec Hausdorff Center for Mathematics, Bonn, Seminar
Dec Rutgers University Seminar
Dec UC Berkeley Colloquium and Number Theory Seminar
Dec Washington University in St. Louis Seminar
Nov UCSD Analysis Seminar and Number Theory Seminar
Nov Georgia Tech Seminar
Nov Indiana University at Bloomington Colloquium
Nov Warwick University Number Theory Seminar
Oct Duke University Seminar
Oct Oxford Junior Number Theory Seminar
Oct Reading University Analysis Seminar
Oct Oxford Number Theory Day (organizer)
Sep SAVANT Synergies and Vistas in Analytic Number Theory, Oxford (attendee)
Jul-Aug Visitor at Princeton University
Jun University of Cologne Number Theory Seminar
Jun Analytic Methods for Diophantine Problems, Gttingen

Seminars and Conferences in 2011
Jul Princeton Summer in Analysis and Geometry, Princeton
Jun Workshop on Oscillatory Integrals, Edinburgh
Jun OxPDE Seminar
May Visitor at Princeton University
May Conference in Honor of Eli Stein's 80th Birthday, Princeton (panel discussion)

Apr Caltech Number Theory Seminar
Apr UCSD Analysis Seminar
Apr UCLA Analysis Seminar
Mar University of Birmingham (UK) Analysis Seminar
Feb Harmonic Analysis Workshop, Edinburgh
Jan University of Bristol Analysis Seminar

Seminars and Conferences in 2010
Dec Cardiff Analysis Seminar
Dec University of Bristol Heilbronn (Number Theory) Seminar
Nov Exponential Sums over Finite Fields, ETH Zurich
Nov UCL Pure Maths Seminar

4.1.E Address of project website

http://people.maths.ox.ac.uk/~piercel/mariecurie.shtml

4.2 Use and dissemination of foreground

Section A

Table A1: List of scientific (peer reviewed) publications

No.	Title	Authors (alphabetical)	Journal	Volume	Publisher	Place	Year	Pages	Identifiers	Open Access provided
1	**Counting rational points on smooth cyclic covers**	D. R. Heath-Brown and L. B. Pierce	Journal of Number Theory	132	Elsevier		2012	1741-1757		yes
2	**Simultaneous prime values of pairs of quadratic forms**	D. R. Heath-Brown and L. B. Pierce	In final stages of preparation for submission							yes
3	**Carleson operators of Radon type with polynomial phase**	L. B. Pierce and P. L. Yung	In final stages of preparation for submission							yes

Table A2: List of dissemination activities

No.	Type of activity	Main Leader	Title	Date	Place	Type of audience	Size of audience	Countries addressed
1	Research seminar	L. B. Pierce	UCL Pure Maths Seminar	Nov 2010	London, UK	Scientific Community (SC)	15	UK
2	Conference	L. B. Pierce	Exponential Sums over Finite Fields, ETH Zurich	Nov 2010	Zurich, Switzerland	SC	50	international
3	Research seminar	L. B. Pierce	University of Bristol Heilbronn (Number Theory) Seminar	Dec 2010	Bristol, UK	SC	30	UK
4	Research seminar	L. B. Pierce	Cardiff Analysis Seminar	Dec 2010	Cardiff, UK	SC	20	UK
5	Research seminar	L. B. Pierce	University of Bristol Analysis Seminar	Jan 2011	Bristol, UK	SC	15	UK
6	Conference	L. B. Pierce	Harmonic Analysis Workshop, Edinburgh	Feb 2011	Edinburgh, UK	SC	30	international
7	Research seminar	L. B. Pierce	University of Birmingham (UK) Analysis Seminar	Mar 2011	Birmingham, UK	SC	15	UK
8	Research seminar	L. B. Pierce	UCLA Analysis Seminar	Apr 2011	Los Angeles, US	SC	15	US
9	Research seminar	L. B. Pierce	UCSD Analysis Seminar	Apr 2011	San Diego, US	SC	15	US
10	Research seminar	L. B. Pierce	Caltech Number Theory Seminar	Apr 2011	Pasadena, US	SC	15	US
11	Panel discussion	L. B. Pierce	Conference in Honor of Eli Stein's 80th Birthday, Princeton	May 2011	Princeton, US	SC	100	international

12	Research seminar	L. B. Pierce	OxPDE Seminar	Jun 2011	Oxford, UK	SC	15	UK
13	Conference	L. B. Pierce	Workshop on Oscillatory Integrals, Edinburgh	Jun 2011	Edinburgh, UK	SC	40	international
14	Conference	L. B. Pierce	Analytic Methods for Diophantine Problems, Göttingen	Jun 2012	Göttingen, Germany	SC	50	international
15	Research seminar	L. B. Pierce	University of Cologne Number Theory Seminar	Jun 2012	Cologne, Germany	SC	5	Germany
16	Conference	L. B. Pierce	Oxford Number Theory Day	Oct 2012	Oxford, UK	SC	15	UK
17	Research seminar	L. B. Pierce	Reading University Analysis Seminar	Oct 2012	Reading, UK	SC	10	UK
18	Research seminar	L. B. Pierce	Oxford Junior Number Theory Seminar	Oct 2012	Oxford, UK	SC	10	UK
19	Research seminar	L. B. Pierce	Duke University Seminar	Oct 2012	Durham, US	SC	30	US
20	Research seminar	L. B. Pierce	Warwick University Number Theory Seminar	Nov 2012	Warwick, UK	SC	25	UK
21	Research seminar	L. B. Pierce	Indiana University at Bloomington Colloquium	Nov 2012	Bloomington US	SC	40	US
22	Research seminar	L. B. Pierce	Georgia Tech Seminar	Nov 2012	Atlanta, US	SC	30	US
23	Research seminar	L. B. Pierce	UCSD Analysis Seminar and Number Theory Seminar	Nov 2012	San Diego, US	SC	30	US
24	Research seminar	L. B. Pierce	Washington University in St. Louis Seminar	Dec 2012	St. Louis, US	SC	30	US
25	Research seminar	L. B. Pierce	UC Berkeley Colloquium and Number Theory	Dec 2012	Berkeley, US	SC	50	US

			Seminar					
26	Research seminar	L. B. Pierce	Rutgers University Seminar	Dec 2012	Piscataway, US	SC	30	US
27	Research seminar	L. B. Pierce	Hausdorff Center for Mathematics, Bonn, Seminar	Dec 2012	Bonn, Germany	SC	30	Germany
28	Conference	L. B. Pierce	Number Theory Seminar, AWM Workshop, Joint Mathematics Meetings, San Diego	Jan 2013	San Diego, US	SC	50	international
29	Research seminar	L. B. Pierce	UW Madison Colloquium	Jan 2013	Madison, US	SC	20	US
30	Research seminar	L. B. Pierce	McGill Special Lecture	Jan 2013	Montreal, Canada	SC	25	Canada
31	Research seminar	L. B. Pierce	MIT Analysis Seminar and Number Theory Seminar	Jan 2013	Boston, US	SC	20	US
32	Research seminar	L. B. Pierce	UMinn Special Lecture	Jan 2013	Minneapolis, US	SC	20	US
33	Research seminar	L. B. Pierce	UI Urbana-Champaign Colloquium	Jan 2013	Urbana-Champaign, US	SC	40	US
34	Research seminar	L. B. Pierce	Caltech Colloquium	Jan 2013	Pasadena, US	SC	20	US

Section B (Confidential[6] or public: confidential information to be marked clearly)
Part B1

The applications for patents, trademarks, registered designs, etc. shall be listed according to the template B1 provided hereafter.

The list should, specify at least one unique identifier e.g. European Patent application reference. For patent applications, only if applicable, contributions to standards should be specified. This table is cumulative, which means that it should always show all applications from the beginning until after the end of the project.

(none)

TEMPLATE B1: LIST OF APPLICATIONS FOR PATENTS, TRADEMARKS, REGISTERED DESIGNS, ETC.

Type of IP Rights[7]:	Confidential Click on YES/NO	Foreseen embargo date dd/mm/yyyy	Application reference(s) (e.g. EP123456)	Subject or title of application	Applicant (s) (as on the application)

[6] Note to be confused with the "EU CONFIDENTIAL" classification for some security research projects.

[7] A drop down list allows choosing the type of IP rights: Patents, Trademarks, Registered designs, Utility models, Others.

Part B2 (none)

Please complete the table hereafter:

Type of Exploitable Foreground[8]	Description of exploitable foreground	Confidential Click on YES/NO	Foreseen embargo date dd/mm/yyyy	Exploitable product(s) or measure(s)	Sector(s) of application[9]	Timetable, commercial or any other use	Patents or other IPR exploitation (licences)	Owner & Other Beneficiary(s) involved
	Ex: New superconductive Nb-Ti alloy			*MRI equipment*	*1. Medical 2. Industrial inspection*	*2008 2010*	*A materials patent is planned for 2006*	*Beneficiary X (owner) Beneficiary Y, Beneficiary Z, Poss. licensing to equipment manuf. ABC*

In addition to the table, please provide a text to explain the exploitable foreground, in particular:

- Its purpose
- How the foreground might be exploited, when and by whom
- IPR exploitable measures taken or intended
- Further research necessary, if any
- Potential/expected impact (quantify where possible)

[19] A drop down list allows choosing the type of foreground: General advancement of knowledge, Commercial exploitation of R&D results, Exploitation of R&D results via standards, exploitation of results through EU policies, exploitation of results through (social) innovation.

[9] A drop down list allows choosing the type sector (NACE nomenclature) : http://ec.europa.eu/competition/mergers/cases/index/nace_all.html

4.3 Report on societal implications

Replies to the following questions will assist the Commission to obtain statistics and indicators on societal and socio-economic issues addressed by projects. The questions are arranged in a number of key themes. As well as producing certain statistics, the replies will also help identify those projects that have shown a real engagement with wider societal issues, and thereby identify interesting approaches to these issues and best practices. The replies for individual projects will not be made public.

A General Information *(completed automatically when **Grant Agreement number** is entered.*	
Grant Agreement Number:	235572
Title of Project:	CIRCLE METHOD
Name and Title of Coordinator:	Prof. Roger Heath-Brown

B Ethics	
1. Did your project undergo an Ethics Review (and/or Screening)? • If Yes: have you described the progress of compliance with the relevant Ethics Review/Screening Requirements in the frame of the periodic/final project reports? Special Reminder: the progress of compliance with the Ethics Review/Screening Requirements should be described in the Period/Final Project Reports under the Section 3.2.2 *'Work Progress and Achievements'*	*0Yes* *No*
2. Please indicate whether your project involved any of the following issues (tick box) :	***YES***
RESEARCH ON HUMANS	
• Did the project involve children?	
• Did the project involve patients?	
• Did the project involve persons not able to give consent?	
• Did the project involve adult healthy volunteers?	
• Did the project involve Human genetic material?	
• Did the project involve Human biological samples?	
• Did the project involve Human data collection?	
RESEARCH ON HUMAN EMBRYO/FOETUS	
• Did the project involve Human Embryos?	
• Did the project involve Human Foetal Tissue / Cells?	
• Did the project involve Human Embryonic Stem Cells (hESCs)?	
• Did the project on human Embryonic Stem Cells involve cells in culture?	
• Did the project on human Embryonic Stem Cells involve the derivation of cells from Embryos?	
PRIVACY	
• Did the project involve processing of genetic information or personal data (eg. health, sexual lifestyle, ethnicity, political opinion, religious or philosophical conviction)?	
• Did the project involve tracking the location or observation of people?	
RESEARCH ON ANIMALS	
• Did the project involve research on animals?	
• Were those animals transgenic small laboratory animals?	
• Were those animals transgenic farm animals?	
• Were those animals cloned farm animals?	

• Were those animals non-human primates?	
RESEARCH INVOLVING DEVELOPING COUNTRIES	
• Did the project involve the use of local resources (genetic, animal, plant etc)?	
• Was the project of benefit to local community (capacity building, access to healthcare, education etc)?	
DUAL USE	
• Research having direct military use	0 Yes 0 No
• Research having the potential for terrorist abuse	

C Workforce Statistics

3. Workforce statistics for the project: Please indicate in the table below the number of people who worked on the project (on a headcount basis).

Type of Position	Number of Women	Number of Men
Scientific Coordinator	0	1
Work package leaders	0	2
Experienced researchers (i.e. PhD holders)	1	0
PhD Students	0	0
Other	0	0

4. How many additional researchers (in companies and universities) were recruited specifically for this project?	0
Of which, indicate the number of men:	0

D Gender Aspects

5. Did you carry out specific Gender Equality Actions under the project?	○ ●	Yes No

6. Which of the following actions did you carry out and how effective were they?

		Not at all effective				Very effective
☐	Design and implement an equal opportunity policy	○	○	○	○	○
☐	Set targets to achieve a gender balance in the workforce	○	○	○	○	○
☐	Organise conferences and workshops on gender	○	○	○	○	○
☐	Actions to improve work-life balance	○	○	○	○	○
○	Other:					

7. Was there a gender dimension associated with the research content – **i.e. wherever people were the focus of the research as, for example, consumers, users, patients or in trials, was the issue of gender considered and addressed?**

○ Yes- please specify

● No

E Synergies with Science Education

8. Did your project involve working with students and/or school pupils (e.g. open days, participation in science festivals and events, prizes/competitions or joint projects)?

○ Yes- please specify

● No

9. Did the project generate any science education material (e.g. kits, websites, explanatory booklets, DVDs)?

○ Yes- please specify

● No

F Interdisciplinarity

10. Which disciplines (see list below) are involved in your project?

● Main discipline[10]: 1.1

○ Associated discipline[10]: | ○ Associated discipline[10]:

G Engaging with Civil society and policy makers

11a Did your project engage with societal actors beyond the research community? *(if 'No', go to Question 14)*	○ ●	Yes No

11b If yes, did you engage with citizens (citizens' panels / juries) or organised civil society (NGOs, patients' groups etc.)?

● No

○ Yes- in determining what research should be performed

○ Yes - in implementing the research

○ Yes, in communicating /disseminating / using the results of the project

[10] Insert number from list below (Frascati Manual).

11c	**In doing so, did your project involve actors whose role is mainly to organise the dialogue with citizens and organised civil society (e.g. professional mediator; communication company, science museums)?**	○ ●	Yes No

12. Did you engage with government / public bodies or policy makers (including international organisations)

- ● No
- ○ Yes- in framing the research agenda
- ○ Yes - in implementing the research agenda
- ○ Yes, in communicating /disseminating / using the results of the project

13a Will the project generate outputs (expertise or scientific advice) which could be used by policy makers?

- ○ Yes – as a **primary** objective (please indicate areas below- multiple answers possible)
- ○ Yes – as a **secondary** objective (please indicate areas below - multiple answer possible)
- ● No

13b If Yes, in which fields?

Agriculture Audiovisual and Media Budget Competition Consumers Culture Customs Development Economic and Monetary Affairs Education, Training, Youth Employment and Social Affairs		Energy Enlargement Enterprise Environment External Relations External Trade Fisheries and Maritime Affairs Food Safety Foreign and Security Policy Fraud Humanitarian aid		Human rights Information Society Institutional affairs Internal Market Justice, freedom and security Public Health Regional Policy Research and Innovation Space Taxation Transport	

13c If Yes, at which level?

- ○ Local / regional levels
- ○ National level
- ○ European level
- ○ International level

H Use and dissemination

14. How many Articles were published/accepted for publication in peer-reviewed journals?	1
To how many of these is open access[11] provided?	● 1
How many of these are published in open access journals?	0
How many of these are published in open repositories?	1
To how many of these is open access not provided?	0
Please check all applicable reasons for not providing open access:	
❑ publisher's licensing agreement would not permit publishing in a repository ❑ no suitable repository available ❑ no suitable open access journal available ❑ no funds available to publish in an open access journal ❑ lack of time and resources ❑ lack of information on open access ❑ other[12]: ……………	

15. How many new patent applications ('priority filings') have been made? *("Technologically unique": multiple applications for the same invention in different jurisdictions should be counted as just one application of grant).*	0

16. Indicate how many of the following Intellectual Property Rights were applied for (give number in each box).	Trademark	0
	Registered design	0
	Other	0

17. How many spin-off companies were created / are planned as a direct result of the project?	0
Indicate the approximate number of additional jobs in these companies:	0

18. Please indicate whether your project has a potential impact on employment, in comparison with the situation before your project:

❑ Increase in employment, or	❑ In small & medium-sized enterprises
❑ Safeguard employment, or	❑ In large companies
❑ Decrease in employment,	☑ None of the above / not relevant to the project
❑ Difficult to estimate / not possible to quantify	

19. For your project partnership please estimate the employment effect resulting directly from your participation in Full Time Equivalent (*FTE = one person working fulltime for a year*) jobs:	*Indicate figure:* 0

[11] Open Access is defined as free of charge access for anyone via Internet.
[12] For instance: classification for security project.

Difficult to estimate / not possible to quantify	☑

I Media and Communication to the general public

20. As part of the project, were any of the beneficiaries professionals in communication or media relations?

○ Yes ● No

21. As part of the project, have any beneficiaries received professional media / communication training / advice to improve communication with the general public?

○ Yes ● No

22 Which of the following have been used to communicate information about your project to the general public, or have resulted from your project?

☐	Press Release	☐	Coverage in specialist press
☐	Media briefing	☐	Coverage in general (non-specialist) press
☐	TV coverage / report	☐	Coverage in national press
☐	Radio coverage / report	☐	Coverage in international press
☐	Brochures /posters / flyers	☑	Website for the general public / internet
☐	DVD /Film /Multimedia	☐	Event targeting general public (festival, conference, exhibition, science café)

23 In which languages are the information products for the general public produced?

☐	Language of the coordinator	☑	English
☐	Other language(s)		

***Question F-10*:** Classification of Scientific Disciplines according to the Frascati Manual 2002 (Proposed Standard Practice for Surveys on Research and Experimental Development, OECD 2002):

FIELDS OF SCIENCE AND TECHNOLOGY

1. NATURAL SCIENCES

1.1 Mathematics and computer sciences [mathematics and other allied fields: computer sciences and other allied subjects (software development only; hardware development should be classified in the engineering fields)]

1.2 Physical sciences (astronomy and space sciences, physics and other allied subjects)

1.3 Chemical sciences (chemistry, other allied subjects)

1.4 Earth and related environmental sciences (geology, geophysics, mineralogy, physical geography and other geosciences, meteorology and other atmospheric sciences including climatic research, oceanography, vulcanology, palaeoecology, other allied sciences)

1.5 Biological sciences (biology, botany, bacteriology, microbiology, zoology, entomology, genetics, biochemistry, biophysics, other allied sciences, excluding clinical and veterinary sciences)

2 ENGINEERING AND TECHNOLOGY

2.1 Civil engineering (architecture engineering, building science and engineering, construction engineering, municipal and structural engineering and other allied subjects)

2.2 Electrical engineering, electronics [electrical engineering, electronics, communication engineering and systems, computer engineering (hardware only) and other allied subjects]

2.3. Other engineering sciences (such as chemical, aeronautical and space, mechanical, metallurgical and materials engineering, and their specialised subdivisions; forest products; applied sciences such as

geodesy, industrial chemistry, etc.; the science and technology of food production; specialised technologies of interdisciplinary fields, e.g. systems analysis, metallurgy, mining, textile technology and other applied subjects)

3. MEDICAL SCIENCES

3.1 Basic medicine (anatomy, cytology, physiology, genetics, pharmacy, pharmacology, toxicology, immunology and immunohaematology, clinical chemistry, clinical microbiology, pathology)

3.2 Clinical medicine (anaesthesiology, paediatrics, obstetrics and gynaecology, internal medicine, surgery, dentistry, neurology, psychiatry, radiology, therapeutics, otorhinolaryngology, ophthalmology)

3.3 Health sciences (public health services, social medicine, hygiene, nursing, epidemiology)

4. AGRICULTURAL SCIENCES

4.1 Agriculture, forestry, fisheries and allied sciences (agronomy, animal husbandry, fisheries, forestry, horticulture, other allied subjects)

4.2 Veterinary medicine

5. SOCIAL SCIENCES

5.1 Psychology

5.2 Economics

5.3 Educational sciences (education and training and other allied subjects)

5.4 Other social sciences [anthropology (social and cultural) and ethnology, demography, geography (human, economic and social), town and country planning, management, law, linguistics, political sciences, sociology, organisation and methods, miscellaneous social sciences and interdisciplinary , methodological and historical S1T activities relating to subjects in this group. Physical anthropology, physical geography and psychophysiology should normally be classified with the natural sciences].

6. HUMANITIES

6.1 History (history, prehistory and history, together with auxiliary historical disciplines such as archaeology, numismatics, palaeography, genealogy, etc.)

6.2 Languages and literature (ancient and modern)

6.3 Other humanities [philosophy (including the history of science and technology) arts, history of art, art criticism, painting, sculpture, musicology, dramatic art excluding artistic "research" of any kind, religion, theology, other fields and subjects pertaining to the humanities, methodological, historical and other S1T activities relating to the subjects in this group]

2. FINAL REPORT ON THE DISTRIBUTION OF THE EUROPEAN UNION FINANCIAL CONTRIBUTION

This report shall be submitted to the Commission within 30 days after receipt of the final payment of the European Union financial contribution.

Report on the distribution of the European Union financial contribution between beneficiaries

Name of beneficiary	Final amount of EU contribution per beneficiary in Euros
1.	
2.	
n	
Total	