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Abstract

In this deliverable we address fundamental notions underlying the evolvability problem of interoperation hubs, which is concerned with how the information in the interoperation hub evolves to incorporate changes in the domain of interest or more generally of the design of the interoperation hub. In the context of the ACSI Abstract Artifact Model (A^3M), where the Semantic Layer is intended to provide a conceptual specification of the static and dynamic aspects of the domain of interest, such changes may lead to inconsistencies of the data in the artifacts managed in the interoperation hub with respect to the constraints in the Semantic Layer. Instead of declaring actions that lead to an inconsistency as non-executable, and rejecting the resulting system state, we propose mechanism to accept such actions by relying on semantics of evolution for Knowledge Bases (KBs).

The Semantic Layer of the A^3M relies on KBs expressed in Description Logics (DLs) of the *DL-Lite* family, and the problem of evolution of KBs expressed in such DLs is largely unexplored. Here, we study different semantics for KB evolution, distinguishing between two broad classes: model-based approaches and formula-based approaches. We compare the different semantics and provide expressibility and inexpressibility results for *DL-Lite* KBs.

Our results establish the necessary foundations for the problem of KB evolution in the context of the formalisms adopted for the Semantic Layer of the A^3M. With these results at hand, we will be able to explore, in the second iteration of this deliverable, how the results on verification of properties expressed in temporal logics over Knowledge and Action Bases (see Deliverable D4.1) and Semantically-Governed Artifact Systems (see Deliverable D2.3) can be extended to take into account evolvability of the interoperation hub according to KB evolution semantics. Here we provide some preliminary considerations in this direction.

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1 Introduction

In this deliverable we address fundamental notions underlying the evolvability problem of interoperation hubs, which is concerned with how the information in the interoperation hub evolves to incorporate changes in the domain of interest or more generally of the design of the interoperation hub. In the context of the ACSI Abstract Artifact Model (A^3M), where the Semantic Layer is intended to provide a conceptual specification of the static and dynamic aspects of the domain of interest, such changes may lead to inconsistencies of the data in the artifacts managed in the interoperation hub with respect to the static constraints in the Semantic Layer. The most simple and straightforward approach to deal with these kinds of inconsistencies is to declare an action that leads to an inconsistent state as non-executable, and consequently to reject the system state that would result from the execution of such action. Instead, we propose here a mechanism to accept such actions by relying on semantics of evolution for Knowledge Bases (KBs) that is based on repair.

In *KB evolution*, one is interested in incorporating into an existing KB new knowledge, such as the one generated as an effect of action execution. In general, the new knowledge is represented by a set of formulas denoting those properties that should be true after the KB has evolved. In the case where the new knowledge would cause the KB (or relevant parts of it) to become unsatisfiable, the addition of the new knowledge requires particular care. Indeed, suitable changes need to be made in the KB so as to avoid the undesirable interaction, e.g., by deleting parts of the KB that conflict with the new knowledge. Different choices are possible, corresponding to different semantics for KB evolution [1, 33, 23, 50, 26, 44].

The Semantic Layer of the A^3M relies on KBs expressed in Description Logics (DLs) of the *DL-Lite* family, and the problem of evolution of KBs expressed in such DLs is largely unexplored. In the first iteration of this deliverable, we study different semantics for KB evolution, distinguishing between two broad classes: *(i) model-based approaches*, in which the result of evolution of a KB with new knowledge is represented through the set of models that are minimally distant from the models of the original KB; and *(ii) formula-based approaches*, in which the result of evolution of a KB is characterized through a set of formulas (i.e., logical assertions) that minimally differ from the formulas of the original KB. We concentrate on KBs expressed in Description Logics (DLs) of the *DL-Lite* family, which is the formalism underlying the Semantic Layer of the A^3M, and which is at the basis of OWL 2 QL, one of the tractable fragments of OWL 2, the recently proposed revision of the Web Ontology Language. We review known model-based approaches (MBAs) and formula-based (FBAs) approaches for evolution of propositional theories. The main focus of our study is evolution on the data level, so-called *ABox evolution*.

For model-based approaches, we provide a general framework that allows us to lift MBAs from the propositional to the DL case. We then exhibit limitations of a number of model-based approaches. In addition we aim at providing a thorough *understanding* of various important aspects of the ABox evolution problem for *DL-Lite* KBs: in which fragments can evolution be captured? What causes the inexpressibility? Which logic is sufficient to express evolution? Can one approximate evolution in *DL-Lite*, and if yes, how? This work provides some understanding of these issues for an important class of MBAs, which cover the case of both update and revision. We describe what causes inexpressibility, and propose techniques (based on what we call prototypes) that help to approximate evolution according to the well-known Winslett's approach, which is inexpressible in *DL-Lite*. We also identify *DL-Lite* fragments for which evolution is expressible, and for these fragments we provide polynomial-time algorithms to compute the result of evolution.

For formula-based approaches, we show that known approaches are problematic for general *DL-Lite* evolution, either due to high complexity of computation, or because the result of such an action of evolution is not expressible in *DL-Lite*. Building upon the insights gained, we propose two novel formula-based approaches for which evolution is expressible in *DL-Lite*. Moreover, both approaches have attractive properties for ABox evolution: evolution results always exist, are unique, and can be computed in polynomial time.

Our results establish the necessary foundations for the problem of KB evolution in the context of the formalisms adopted for the Semantic Layer of the A^3M. With these results at hand, we will be able to explore, in the second iteration of this deliverable, how the results on verification of properties expressed in temporal logics over Knowledge and Action Bases (see Deliverable D2.4 [8]) and Semantically-Governed Artifact Systems (see Deliverable D2.3 [13]) can be extended to take into account evolvability of the interoperation hub according to repair-based semantics for KB evolution. Here we provide some preliminary considerations in this direction.

1.1 Evolution of Knowledge Bases

Knowledge Bases (KBs) written in the Web Ontology Language (OWL) [30] and its revision OWL 2 [19] are widely used in applications. The formal underpinning of OWL is based on Description Logics (DLs) – knowledge representation formalisms with well-understood computational properties [4]. A DL KB $\mathcal{K}$ consists of a TBox $\mathcal{T}$, describing schema-level domain knowledge, and an ABox $\mathcal{A}$, providing data about specific individuals.

Traditionally DLs have been used for modeling at the intensional level the *static* and structural aspects of application domains [4]. Recently, however, the scope of KBs has broadened, and they are now used also for providing support in the maintenance and *evolution* phase of information systems. This makes it necessary to study *evolution of Knowledge Bases* from both foundational and practical perspectives [27, 29, 45, 16, 49, 31, 39, 28, 21].

The goal of KB evolutions is to incorporate new knowledge $\mathcal{N}$ into an existing KB $\mathcal{K}$ so as to take into account changes that occur in the underlying application domain. The impact of such incorporation on the semantics of $\mathcal{K}$x is difficult to predict and understand. In general, $\mathcal{N}$ is represented by a set of formulas denoting properties that should be true after $\mathcal{K}$ has evolved, and the result of evolution, denoted $\mathcal{K} \diamond \mathcal{N}$, is also intended to be a set of formulas. In the case where $\mathcal{N}$ interacts with $\mathcal{K}$ in an undesirable way, e.g., by causing the KB to become unsatisfiable, the new knowledge $\mathcal{N}$ cannot be simply added to the KB. Instead, suitable changes need to be made in $\mathcal{K}$ so as to avoid this undesirable interaction, e.g., by deleting parts of $\mathcal{K}$ conflicting with $\mathcal{N}$. Different choices for changes are possible, corresponding to different approaches to semantics for KB evolution [1, 33, 23]. In our work we focus on evolution which covers both *knowledge revision* and *update* [2, 42].

From a logic-based perspective, the desirable properties of evolution are dictated by the *principle of minimal change* [2], according to which the semantics of the KB should change "as little as possible", thus ensuring that the evolution has the least possible impact. Logic-based semantics derived from the principle of minimal change can be divided into two main families of approaches: *model-based* approaches (MBAs) or *formula-based* approaches (FBAs). Under both types of semantics, evolution of $\mathcal{K}$ with respect to $\mathcal{N}$ results in another KB $\mathcal{K}'$ in which the required information is incorporated, retracted, or updated; the difference is in the way $\mathcal{K}'$ is obtained.

Under MBAs the models $\mathcal{M}$ of $\mathcal{K}$ (i.e., the set of all first order interpretations satisfying $\mathcal{K}$) evolves into a set $\mathcal{M}' = \mathcal{K} \diamond \mathcal{N}$ of interpretations that are "as close as possible" to those in $\mathcal{M}$ (w.r.t. some notion of distance between interpretations); then, $\mathcal{K}'$ is the KB that axiomatises $\mathcal{K} \diamond \mathcal{N}$ [45, 16, 49, 34, 49]. Model-based evolution of KBs where both $\mathcal{K}$ and $\mathcal{N}$ are expressed in a language of the *DL-Lite* family [11] has recently received a lot of attention [41, 22, 49]. The focus on *DL-Lite* is not surprising since this family has been specifically designed to capture the fundamental constructs of widely used conceptual modeling formalisms, such as UML Class Diagrams and the Entity-Relationship model [9, 12]. *DL-Lite* is also at the basis of OWL 2 QL, one of the tractable fragments (or profiles) of the OWL 2 standard [47, 19].

In [16, 51, 17, 18] we introduce eight natural model-based semantics and show that one can find *DL-Lite* KBs $\mathcal{K}$ and $\mathcal{N}$ such that $\mathcal{K} \diamond \mathcal{N}$ cannot be expressed in *DL-Lite*, that is, *DL-Lite* is *not closed* under MBA to evolution. In this report we review some of these results. Furthermore, in [34, 36, 35, 37] we make an attempt to gain a thorough *understanding* of:

(1) DL-Lite w.r.t. evolution: Which fragments of *DL-Lite* are closed under model-based semantics of evolution? Which *DL-Lite* formulas are responsible for inexpressibility of model-based

semantics of evolution?

(2) Evolution w.r.t. DL-Lite: Is it possible to capture evolution of *DL-Lite* KBs in richer logics and how can it be done? Which are these logics?

(3) Approximation of evolution results: For *DL-Lite* KBs $\mathcal{K}$ and $\mathcal{N}$, is it possible to obtain "good" approximations of $\mathcal{K} \diamond \mathcal{N}$ in *DL-Lite* and how can it be done?

In this report we focus mostly on these Items (1)-(3).

Under FBAs, $\mathcal{K}'$ is a finite subset of the *deductive closure* of $\mathcal{K}$ satisfying the evolution requirements, with FBS differing in their subset selection mechanism. FBAs have been less studied in the context of KBs [16, 40].

For both MBAs and FBAs we consider the following scenario, which is important for many KB design and management tasks:

> *ABox evolution*, where $\mathcal{N}$ is a set of ABox assertion, and the TBox of the original KB cannot be modified as a result of the evolution.

ABox evolution is important for applications relying on widely-used *reference TBoxes.* For example, experimental results on gene extraction can be described using an ABox according to standard gene TBoxes. New experiments may imply that facts about specific genes no longer hold, which should be reflected in the ABox; at the same time, TBoxes should clearly not be affected by these manipulations of the data. Another significant area where ABox evolution is particularly relevant is Semantic Web services, where one is interested in studying the effects of services that perform operations over the instance data. Such data are inherently incomplete, and thus can be effectively represented by means of an ABox. Moreover, the services have to obey the semantics of the domain of interest, which is modeled through a TBox, which is assumed not to change over time.

ABox evolution will be the key topic discussed throughout the report. We now discuss how it is related to ACSI.

1.2 Structure of the Deliverable

The rest of the deliverable is structured as follows:

- In Section 2, we introduce the DLs on which we base our results on KB evolution. Specifically, in Section 2.1 we introduce the DL *DL-Lite*$_{core}$, the basic member of the *DL-Lite* family of lightweight DLs. In Section 2.2 we introduce *DL-Lite*^{pr}, a restriction on *DL-Lite*$_{core}$, which is interesting in practice because, on the one hand, it extends the first-order fragment of the RDFS ontology language [48], and, on the other hand, we prove that it is closed under most of MBAs and for the other MBAs "good" approximations of evolution can be efficiently computed.
- In Section 3, we discuss model-based evolution semantics.
  - In Section 3.1 we introduce a three-dimensional space of evolution semantics eight semantics of this space, and the two main problems related to evolution.
  - In Section 3.2 we study evolution of *DL-Lite*^{pr} KBs under three model-based semantics. More precisely, we prove that
    - *DL-Lite*^{pr} is closed under two of them and present two corresponding polynomial-time algorithms to compute evolution results (in Sections 3.2.1 and 3.2.2);
    - *DL-Lite*^{pr} is not closed under the third semantics, and for this case we present a polynomial-time approximation algorithm.

  - In Section 3.3 we introduce the notion of *subsumption relation* between evolution semantics and prove this relation between some model-based semantics, first, for arbitrary DLs, and then for *DL-Lite*^{pr}. In particular, we show that for *DL-Lite*^{pr} all the eight MBAs considered in this paper collapse into three equivalence classes w.r.t. the subsumption relation. Moreover, the three MBAs for which we study the evolution in Section 3.2 *(i)* are not equivalent to each other w.r.t. the subsumption relation and *(ii)* are representatives for these equivalence classes. Thus, the results we present in Section 3.2 carry out to all the other model-based semantics of this work.
  - In Section 3.4 we study evolution of the full *DL-Lite*$_{core}$ under an important MBA corresponding to the well accepted *Winslett's semantics* [50], which is one of the eight MBAs considering in the paper.
    * For this semantics we show which combination of TBox and ABox assertions in $\mathcal{K}$ together with ABox assertions of $\mathcal{N}$ leads to inexpressibility of evolution (Section 3.4.1).
    * We introduce *prototypes*, which are a generalization of canonical models.
  - In Section 3.5 we continue the study of Section 3.4 and show how one can efficiently approximate evolution under this semantics.

- In Section 4 we review known classical formula-based semantics and introduce Bold Semantics. In Section 4.1, we study properties of Bold Semantics. We prove uniqueness of ABox evolution under Bold Semantics and present an efficient algorithm to compute it. Then, in Section 4.2, we discuss drawbacks of Bold Semantics and introduce Careful Semantics, which addresses the drawbacks. We prove uniqueness of ABox evolution under Careful Semantics and present an efficient algorithm to compute it.

- In Section 5, we provide some preliminary considerations on how we can apply repair-based semantics of KB evolution, such as those studied in Sections 3 and 4 to the setting where the knowledge to be evolved is the data component of a Knowledge and Action Base (KAB) of an Artifact Based System.

- In Section 6 we draw some conclusion and outline how the work in ACSI Task 2.5 will be continued in the 3rd year of the project.

We moved some proofs of Section 3 to an appendix to improve readability of the report.

1.3 Publications

Our results on KB evolution have already given rise to several publications: [16, 34, 17, 51, 36, 35]. A version of our work with full proofs has been submitted to an international journal [38]. In addition, we have presented a tutorial on KB evolution at an international workshop in May 2012 [20]. We presented our initial results on how to apply KB evolution based on repair semantics to Data Centric Business Processes as a statement of purpose paper at an international workshop [14].

2 Description Logics *DL-Lite* and Knowledge Evolution

We first present the Description Logics *DL-Lite*$_{core}$ and its fragment *DL-Lite*^{pr}. We then define the problem of knowledge evolution.

2.1 The Description Logic *DL-Lite*$_{core}$

In DLs [4], the domain of interest is modeled by means of *concepts*, denoting sets of objects, *roles*, denoting binary relations between objects, and *constants*, denoting objects. Complex concepts and roles are obtained from atomic ones by applying suitable constructs. We consider here the logic *DL-Lite*$_{core}$, a member of the *DL-Lite* family of lightweight DLs. All other members of the family extend *DL-Lite*$_{core}$ with some constructs (see [11, 3] for details), but these extra constructs are not important for the results of this paper. In *DL-Lite*$_{core}$, (complex) concepts and roles are constructed according to the following syntax:

$$B ::= A \mid \exists R, \qquad C ::= B \mid \neg B, \qquad R ::= P \mid P^-,$$

where A denotes an *atomic concept*, B a *basic concept*, and C a *general* concept; P denotes an *atomic role* and R a *basic* role, i.e., either a direct or inverse role. Note that further in the paper, when we write R^- for a basic role R, it will signify *(i)* P^- if $R = P$, or *(ii)* P if $R = P^-$.

In *DL-Lite*$_{core}$ the knowledge about the domain of interest is represented by means of a *knowledge base* (KB) $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, constituted by a set of assertions partitioned into a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. A TBox consists of *positive* and *negative (concept) inclusion assertions* (PIs and NIs for short) of the form

$$B_1 \sqsubseteq B_2, \qquad B_1 \sqsubseteq \neg B_2,$$

which are used to assert intensional domain knowledge. Further in this paper, if $B_1 \sqsubseteq \neg B_2 \in \mathcal{T}$, we will say that B_1 and B_2 are *disjoint*. An ABox consists of *membership assertions* (MAs) of the form

$$P(c, d), \qquad \exists R(c), \qquad A(c), \qquad \neg A(c),$$

where c and d are constants. MAs of the third form are called *negative*, while the rest of MAs called *positive*. An *atom* is an expression of the form $A(a)$ or $P(a, b)$ for A an atomic concept, P an atomic role, and a and b constants. A *literal* is an atom or the negation of an atom, that is, $A(a)$ or $\neg A(a)$.

Example 2.1. Consider a KB $\mathcal{K}_0 = \mathcal{T}_0 \cup \mathcal{A}_0$. TBox $\mathcal{T}_0$:

$$\mathcal{T}_0 = \{\mathit{Card} \sqsubseteq \mathit{Priest},\ \exists \mathit{HasHusb}^- \sqsubseteq \neg \mathit{Priest},\ \mathit{Husb} \sqsubseteq \exists \mathit{HasHusb}^-,\ \mathit{Wife} \sqsubseteq \exists \mathit{HasHusb}\}.$$

Intuitively, $\mathcal{T}_0$ says that cardinals are priests, husbands are not priests, wives and husbands are those who participate in the *HasHusb* relationship. Consider an ABox $\mathcal{A}_0$:

$$\mathcal{A}_0 = \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan}), \mathit{Husb}(\mathit{john}), \mathit{Wife}(\mathit{mary}), \mathit{Wife}(\mathit{chloe}), \mathit{HasHusb}(\mathit{mary}, \mathit{john})\}.$$

Intuitively, $\mathcal{A}_0$ says that there are two priests *pedro* and *ivan*, one husband *john*, two wives *mary* and *chloe*, and *mary* and *john* are married. ■

A signature Σ is a finite set of concept and role names and constants. The signature $\Sigma(F)$ of an assertion F is the set of concept and role names and constants occurring in F, and the signature of a KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ is $\Sigma(\mathcal{K}) = \bigcup_{\mathcal{F} \in \mathcal{T} \cup \mathcal{A}} \Sigma(F)$. The size $|\mathcal{K}|$ of a KB $\mathcal{K}$ is the size $|\Sigma(\mathcal{K})|$ of its signature. We say that a KB $\mathcal{K}$ is over a signature Σ_0 if $\Sigma(\mathcal{K}) \subseteq \Sigma_0$. The *active domain* of $\mathcal{K}$, denoted $\mathsf{adom}(\mathcal{K})$, is the set of all constants occurring in $\mathcal{K}$. Note that $\mathsf{adom}(\mathcal{K}) \subseteq \Sigma(\mathcal{K})$.

The semantics of *DL-Lite*$_{core}$ KBs is given in the standard way, using first order interpretations, which we assume to be all over the same countable domain Δ. An *interpretation* $\mathcal{I}$ *over a signature* Σ_0 (or just *interpretation* when the signature is clear from the context or not important) is a function $\cdot^\mathcal{I}$ that assigns to each concept C a subset $C^\mathcal{I}$ of Δ, and to each role R a binary relation $R^\mathcal{I}$ over Δ in such a way that $A^\mathcal{I} \subseteq \Delta$ (resp., $P^\mathcal{I} \subseteq \Delta \times \Delta$) for every $A \in \Sigma_0$ (resp., for every $P \in \Sigma_0$), $(\neg B)^\mathcal{I} = \Delta \backslash B^\mathcal{I}$, $(\exists R)^\mathcal{I} = \{a \mid \exists a'.(a, a') \in R^\mathcal{I}\}$, and $(P^-)^\mathcal{I} = \{(a_2, a_1) \mid (a_1, a_2) \in P^\mathcal{I}\}$. It is a common practice in the DL community to assume that *DL-Lite* is under so-called *standard names* assumption [11, 3], that is, Δ contains the constants and for every interpretation $\mathcal{I}$ it holds that $c^\mathcal{I} = c$. Hereafter we also adopt this assumption. It is opened how our results can be extended for the case of *DL-Lite*$_{core}$ without standard names, though it seems to be harmless for the results of this work.

An interpretation $\mathcal{I}$ is a *model* of an inclusion assertion $D_1 \sqsubseteq D_2$ if $D_1^{\mathcal{I}} \subseteq D_2^{\mathcal{I}}$, of a membership assertion $A(a)$ (resp., $\neg A(a)$) if $a \in A^{\mathcal{I}}$ (resp., $a \notin A^{\mathcal{I}}$), and of $P(a, b)$ if $(a, b) \in P^{\mathcal{I}}$. It is often convenient to view interpretations as sets of atoms and say that $A(a) \in \mathcal{I}$ iff $a \in A^{\mathcal{I}}$ and $P(a, b) \in \mathcal{I}$ iff $(a, b) \in P^{\mathcal{I}}$. If, under this view, $\mathcal{I}' \subseteq \mathcal{I}$, we say that $\mathcal{I}'$ is a *submodel* of $\mathcal{I}$. As usual, we use $\mathcal{I} \models F$ to denote that $\mathcal{I}$ is a model of an assertion F and $\mathcal{I} \models \mathcal{K}$ to denote that $\mathcal{I}$ is an interpretation over $\Sigma(\mathcal{K})$ and $\mathcal{I} \models F$ for each assertion F of a KB $\mathcal{K}$. We use $\mathsf{Mod}(\mathcal{K})$ to denote the set of all models of $\mathcal{K}$. A KB is *satisfiable* if it has at least one model. Further in the paper we consider *only* satisfiable KBs. This assumption does not affect the complexity results of this paper due to nice computational properties of the *DL-Lite* family, in particular, w.r.t. the size of the extensional information (i.e., the data in the ABox). For example, KB satisfiability has polynomial-time complexity in the size of the TBox and logarithmic-space complexity[1] in the size of the ABox [3, 43].

We use *entailment between KBs*, denoted $\mathcal{K} \models \mathcal{K}'$, in the standard sense, i.e., every model of $\mathcal{K}$ is also a model of $\mathcal{K}'$ (similarly we use entailment between TBoxes and between ABoxes). An ABox $\mathcal{A}$ $\mathcal{T}$*-entails* an ABox $\mathcal{A}'$, denoted $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}'$, if $\mathcal{T} \cup \mathcal{A} \models \mathcal{A}'$, and $\mathcal{A}$ is $\mathcal{T}$*-equivalent* to $\mathcal{A}'$, denoted $\mathcal{A} \equiv_{\mathcal{T}} \mathcal{A}'$, if $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}'$ and $\mathcal{A}' \models_{\mathcal{T}} \mathcal{A}$. We say that $\mathcal{A}$ and $\mathcal{A}'$ are $\mathcal{T}$*-satisfiable* if $\mathcal{A} \cup \mathcal{A}' \not\models_{\mathcal{T}} \bot$, i.e., it does not imply falsehood (denoted using $\bot$).

The deductive *closure of a TBox* $\mathcal{T}$, denoted $\mathsf{cl}(\mathcal{T})$, is the set of all TBox assertions F such that $\mathcal{T} \models F$. For a satisfiable KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, the *closure of* $\mathcal{A}$ (w.r.t. $\mathcal{T}$), denoted $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$, is the set of all membership assertions f (both positive and negative) over $\mathsf{adom}(\mathcal{K})$ such that $\mathcal{A} \models_{\mathcal{T}} f$. In *DL-Lite*$_{core}$ both $\mathsf{cl}(\mathcal{T})$ and $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ are computable in time quadratic in the number of assertions of $\mathcal{T}$ and $\mathcal{T} \cup \mathcal{A}$, respectively. Whenever needed, we will assume w.l.o.g. that all TBoxes and ABoxes are closed.

A *homomorphism* μ from an interpretation $\mathcal{I}$ to an interpretation $\mathcal{J}$ over the same signature Σ_0, is a structure-preserving mapping from Δ to Δ satisfying: *(i)* $\mu(a) = a$ for every constant $a \in \Sigma_0$; *(ii)* if $x \in A^{\mathcal{I}}$ (resp., $(x, y) \in P^{\mathcal{I}}$), then $\mu(x) \in A^{\mathcal{J}}$ (resp., $(\mu(x), \mu(y)) \in P^{\mathcal{J}}$) for every atomic concept A (resp., atomic role P). We write $\mathcal{I} \hookrightarrow \mathcal{J}$ if there is a homomorphism from $\mathcal{I}$ to $\mathcal{J}$.

Now we define the notion of *chase* of an ABox from [11], which is an adaptation of the notion of *restricted chase* from [32]. Let $\mathcal{T}$ and $\mathcal{A}$ be a *DL-Lite*$_{core}$ TBox and ABox, respectively. Then, a *chase* of $\mathcal{A}$ w.r.t. $\mathcal{T}$, denoted $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, is an interpretation of $\mathcal{T} \cup \mathcal{A}$ that can be defined procedurally as follows: take $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \{A(a), P(b, c) \mid A(a), P(b, c) \in \mathcal{A}\}$ and apply the following rules until none of them is applicable.

- if $A_1(x) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, $A_1 \sqsubseteq A_2 \in \mathcal{T}$, and $A_2(x) \notin \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, then $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \mathsf{chase}_{\mathcal{T}}(\mathcal{A}) \cup \{A_2(x)\}$;
- if $A(x) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, $A \sqsubseteq \exists R \in \mathcal{T}$, and for any y it holds $R(x, y) \notin \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, then $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \mathsf{chase}_{\mathcal{T}}(\mathcal{A}) \cup \{R(x, y_{new})\}$, where y_{new} is a fresh element that has not appeared in $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ before;
- if $R(x, y) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, $\exists R \sqsubseteq A \in \mathcal{T}$, and $A(x) \notin \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, then $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \mathsf{chase}_{\mathcal{T}}(\mathcal{A}) \cup \{A(x)\}$;
- if $R_1(x, z) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, $\exists R_1 \sqsubseteq \exists R_2 \in \mathcal{T}$, and for any y it holds $R_2(x, y) \notin \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, then $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \mathsf{chase}_{\mathcal{T}}(\mathcal{A}) \cup \{R(x, y_{new})\}$, where y_{new} is a fresh element that has not appeared in $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ before.

Note, that the procedure does not in general terminate and $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ can be an infinite interpretation. It was shown in [11] that $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ is a *canonical model* for $\mathcal{T} \cup \mathcal{A}$, that is, it can be homomorphically embedded into every model of $\mathcal{T} \cup \mathcal{A}$.[2] Further, we will refer to $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ as $\mathcal{I}_{\mathsf{can}}$ assuming that $\mathcal{K}$ is clear from the context. We also can naturally extend the notion of

[1] Actually, the data complexity of satisfiability and of other inference tasks that involve the ABox is AC^0.

[2] Note that in data-exchange this notion of canonical models corresponds to so-called *core solutions* of data-exchange settings [25]. The difference between canonical models and core solutions is that the former ones can be infinite, while the later ones are always finite.

chase $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ from ABoxes $\mathcal{A}$ to interpretations $\mathcal{I}$ i.e., to $\mathsf{chase}_{\mathcal{T}}(\mathcal{I})$, since an interpretation can be seen as an infinite ABox.

Finally, we recall that *DL-Lite*$_{core}$ has the finite model property, i.e., every satisfiable *DL-Lite*$_{core}$ KB has at least one finite model (this follows from results of [46]).

2.2 The Description Logic *DL-Lite*^{pr}

In this section we introduce a restriction of *DL-Lite*$_{core}$, which we call *DL-Lite*^{pr} (*pr* stands for *positive role* interaction). Intuitively, in *DL-Lite*^{pr} "negative" information on both ABox and TBox level that involves roles is forbidden. More precisely:

Definition 2.1 (The DL *DL-Lite*^{pr})**.** A *DL-Lite*$_{core}$ knowledge base $\mathcal{K}$ is in *DL-Lite*^{pr} if for every basic role R *none* of the following two entailments holds:

$$\textit{(i)}\ \mathcal{K} \models \neg\exists R(a) \quad \text{and} \quad \textit{(ii)}\ \mathcal{K} \models \exists R \sqsubseteq \neg B, \tag{1}$$

where B is some basic concept and a is some constant.

Example 2.2. Consider again $\mathcal{K}_0$ of Example 2.1. To see that $\mathcal{K}_0$ is not in *DL-Lite*^{pr}, observe that it violates both Case (i) and Case (ii) of Equation (1). Indeed, $\mathcal{K}_0 \models \neg\exists \mathit{HasHusb}^-(a)$ for $a \in \{\mathit{pedro}, \mathit{ivan}\}$ and thus Case (i) is violated. Moreover, $\mathcal{T}_0 \models \exists \mathit{HasHusb}^- \sqsubseteq \neg \mathit{Priest}$ and thus Case (ii) is violated. Consider the following subset of $\mathcal{T}_0$:

$$\mathcal{T}_1 = \{\mathit{Card} \sqsubseteq \mathit{Priest},\ \mathit{Husb} \sqsubseteq \neg \mathit{Priest}\}. \tag{2}$$

Clearly $\mathcal{T}_1$ is in *DL-Lite*^{pr}. ■

DL-Lite^{pr} is defined semantically, but one can syntactically check in polynomial time whether a given *DL-Lite*$_{core}$ KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ is in *DL-Lite*^{pr}. Condition (i) of Definition 2.2 can be checked by computing (in time polynomial in $|\mathcal{K}|$) the closure $\mathsf{cl}(\mathcal{K})$ of $\mathcal{K}$ and verifying whether an assertion of the form $\neg\exists R(a)$ is in the closure. Condition (ii) can be verified (in time quadratic in the size $|\mathcal{T}|$) by computing the closure $\mathsf{cl}(\mathcal{T})$ of $\mathcal{T}$, and checking that no assertion of the form $\exists R \sqsubseteq \neg B$ is in the closure.

We see *DL-Lite*^{pr} as an important language to study because it is an extension of the RDFS ontology language [48] (more precisely, of the first-order logic fragment of RDFS). *DL-Lite*^{pr} adds to RDFS the ability of expressing disjointness of atomic concepts ($A_1 \sqsubseteq \neg A_2$) and mandatory participation to roles ($A \sqsubseteq \exists R$).

2.3 Evolution of Knowledge Bases

We introduce now formally the problem of ABox evolution of DL knowledge bases.

Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL KB and $\mathcal{N}$ a "new" ABox. Intuitively, $\mathcal{N}$ represents the information that is considered to be true and we study how to incorporate assertions of $\mathcal{N}$ into $\mathcal{K}$, that is, how $\mathcal{K}$ *evolves* w.r.t. $\mathcal{N}$ [27]. In detail, we study *evolution operators* $\diamond$ that take $\mathcal{K}$ and $\mathcal{N}$ as input and return, preferably in *polynomial time*, a DL KB $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$ (with the same TBox as that of $\mathcal{K}$) that captures the evolution, and that we call *the (ABox) evolution of* $\mathcal{K}$ *w.r.t.* $\mathcal{N}$. Based on the evolution principles of [16], we require $\mathcal{K}$ and $\mathcal{K}'$ to be satisfiable. Now we define evolution settings.

Definition 2.2 (Evolution Setting)**.** Let $\mathcal{D}$ be a DL. A $\mathcal{D}$*-evolution setting* (or just *evolution setting, when* $\mathcal{D}$ *is clear*) is a tuple $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{N}$ a $\mathcal{D}$ ABox such that $\mathcal{N}$ contains only positive MAs, and both $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{T} \cup \mathcal{N}$ are satisfiable $\mathcal{D}$ KBs.[3]

In this paper we will focus on *DL-Lite*$_{core}$ (resp., *DL-Lite*^{pr}) evolution settings.

[3] Satisfiability of $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{T} \cup \mathcal{N}$ is dictated by the evolution principles of [16].

Example 2.3. Consider the KB $\mathcal{K}_0$ from Example 2.1 and two following ABoxes:

$$\mathcal{N}_1 = \{Priest(john)\} \qquad \text{and} \qquad \mathcal{N}_2 = \{Husb(pedro),\ Wife(tanya)\}. \tag{3}$$

The pairs $(\mathcal{K}_0, \mathcal{N}_1)$ and $(\mathcal{K}_0, \mathcal{N}_2)$ are *DL-Lite$_{core}$*-evolution settings. ■

Definition 2.3. An *evolution* for a $\mathcal{D}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ can now be defined as a $\mathcal{D}$ KB $\mathcal{K}'$ that *(i)* preserves $\mathcal{N}$ and *(ii)* changes the semantics of $\mathcal{K}$ "as little as possible", due to the *principle of minimal change.*

There are two main families of approaches to address the principle of minimal change: *model-based approaches* (MBAs) and *formula-based approaches* (FBAs).

3 Model-Based Evolution

3.1 Local and Global Evolution Semantics

We introduce now formally the problem of ABox evolution of DL knowledge bases, concentrating on model-based approaches. We discuss different semantics for the problem and put them into relationship with each other. Specifically, we focus on the eight semantics that have been presented first in [16], and that are the result of considering the problem space according to three orthogonal dimensions (see Figure 1, right). Notice that the notions we introduce do not depend on any specific DL, although we will apply them later to the case of *DL-Lite$_{core}$* and *DL-Lite^{pr}*.

Under *model-based approaches* (MBAs), these two conditions on $\mathcal{K}'$ are reflected as follows: the set of models $\mathsf{Mod}(\mathcal{K}')$ of $\mathcal{K}'$ is precisely the set of interpretations $\mathcal{J}$ such that *(i)* $\mathcal{J} \models \mathcal{T} \cup \mathcal{N}$ and *(ii)* $\mathcal{J}$ is "minimally distant" from the models of $\mathcal{K}$.

Katsuno and Mendelzon [33] considered two ways, so-called *local* and *global*, of deciding which models are minimally distant from the models of $\mathcal{K}$ (w.r.t. some notion of distance), where the former choice corresponds to *knowledge update* and the latter one to *knowledge revision.* Now we discuss these two ways in more details.

The idea of the local approaches is to consider all models of $\mathcal{K}$ one by one, and for each model $\mathcal{I}$ to take those models $\mathcal{J}$ of $\mathcal{T} \cup \mathcal{N}$ that are minimally distant from $\mathcal{I}$. Formally,

Definition 3.1 (Local MBA)**.** Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a $\mathcal{D}$-evolution setting, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, and let $\mathsf{dist}(\cdot,\cdot)$ be a distance function between interpretations. For $\mathcal{I}$ an interpretation, let $\mathsf{loc\_min}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ be the set of interpretations $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{T} \cup \mathcal{N}$ and on $\mathcal{J}$ the value of $\mathsf{dist}(\mathcal{I}, \mathcal{J})$ is minimal for the given $\mathcal{I}$. Then, $\mathcal{K}' \in \mathcal{D}$ is $\mathbf{L}$-*evolution* for $\mathcal{E}$ if $\mathsf{Mod}(\mathcal{K}') = \mathcal{K} \diamond_{\mathbf{L}} \mathcal{N}$, where

$$\mathcal{K} \diamond_{\mathbf{L}} \mathcal{N} = \bigcup_{\mathcal{I} \models \mathcal{K}} \mathsf{loc\_min}(\mathcal{I}, \mathcal{T}, \mathcal{N}).$$

The distance function dist varies from approach to approach and commonly takes as values either numbers or subsets of some fixed set. We will have a discussion on distance functions later in this section.

Example 3.1. To get a better intuition of the local semantics, consider Figure 1, left, where we present two models $\mathcal{I}_0$ and $\mathcal{I}_1$ of a KB $\mathcal{K}$ and four models $\mathcal{J}_0, \ldots, \mathcal{J}_3$ of $\mathcal{T} \cup \mathcal{N}$. We represent the distance between a model of $\mathcal{K}$ and a model of $\mathcal{T} \cup \mathcal{N}$ by the length of the line connecting them. Solid lines correspond to minimal distances, while dashed lines to distances that are not minimal. In this figure, $\mathsf{loc\_min}(\mathcal{I}_0, \mathcal{T}, \mathcal{N}) = \{\mathcal{J}_0\}$ and $\mathsf{loc\_min}(\mathcal{I}_1, \mathcal{T}, \mathcal{N}) = \{\mathcal{J}_2, \mathcal{J}_3\}$. ■

Under a global approach, one chooses models of $\mathcal{T} \cup \mathcal{N}$ that are minimally distant from the "whole" $\mathcal{K}$:

Definition 3.2 (Global MBA)**.** Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a $\mathcal{D}$-evolution setting and $\mathsf{dist}(\cdot,\cdot)$ a distance function between interpretations. For an interpretation $\mathcal{J}$, let $\mathsf{dist}(\mathsf{Mod}(\mathcal{K}), \mathcal{J}) =$

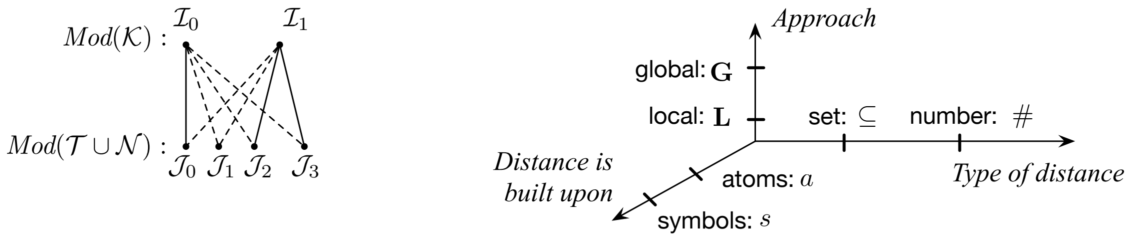

**Figure 1 – Left: measuring distances between models and finding local minimums (Example 3.1).
Right: three-dimensional space of approaches to model-based evolution semantics.**

$\min_{\mathcal{I} \in \mathsf{Mod}(\mathcal{K})} \mathsf{dist}(\mathcal{I}, \mathcal{J})$. Furthermore, let $\mathsf{glob\_min}(\mathcal{K}, \mathcal{N})$ be the set of interpretations $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{N}$ and for each interpretation $\mathcal{J}'$ such that $\mathcal{J}' \models \mathcal{N}$, it *does not hold* that $\mathsf{dist}(\mathsf{Mod}(\mathcal{K}), \mathcal{J}') < \mathsf{dist}(\mathsf{Mod}(\mathcal{K}), \mathcal{J})$. Then, $\mathcal{K}' \in \mathcal{D}$ is $\mathbf{G}$-*evolution* for $\mathcal{E}$ if $\mathsf{Mod}(\mathcal{K}') = \mathcal{K} \diamond_{\mathbf{G}} \mathcal{N}$, where

$$\mathcal{K} \diamond_{\mathbf{G}} \mathcal{N} = \mathsf{glob\_min}(\mathcal{K}, \mathcal{N}).$$

Example 3.2. Consider again Figure 1, left, and assume that the distance between $\mathcal{I}_0$ and $\mathcal{J}_0$ is the global minimum. Thus, we obtain that $\mathsf{glob\_min}(\mathcal{K}, \mathcal{N}) = \{\mathcal{J}_0\}$. ■

Measuring Distance Between Interpretations The classical MBAs were developed for propositional theories [23], where interpretations can be seen as finite sets of propositional symbols. In that case, two distance functions have been introduced, respectively based on symmetric difference "$\ominus$" and on the cardinality of symmetric difference:

$$\mathsf{dist}_{\subseteq}(\mathcal{I}, \mathcal{J}) = \mathcal{I} \ominus \mathcal{J} \qquad \text{and} \qquad \mathsf{dist}_{\#}(\mathcal{I}, \mathcal{J}) = |\mathcal{I} \ominus \mathcal{J}|, \tag{4}$$

where $\mathcal{I} \ominus \mathcal{J} = (\mathcal{I} \setminus \mathcal{J}) \cup (\mathcal{J} \setminus \mathcal{I})$. Distances under $\mathsf{dist}_{\subseteq}$ are sets and are compared by set inclusion, that is, $\mathsf{dist}_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \leq \mathsf{dist}_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2)$ if and only if $\mathsf{dist}_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \subseteq \mathsf{dist}_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2)$. Finite distances under $\mathsf{dist}_{\#}$ are natural numbers and are compared in the standard way.

These distances can be extended to DL interpretations in two ways. First, one can consider interpretations $\mathcal{I}$ and $\mathcal{J}$ as sets of atoms, in which case the symmetric difference $\mathcal{I} \ominus \mathcal{J}$ and the corresponding distances are defined as in the propositional case. In contrast to the propositional case, however, $\mathcal{I} \ominus \mathcal{J}$ (and hence also distances) can be infinite. We denote the distances in Equation (4) as $\mathsf{dist}^a_{\subseteq}(\mathcal{I}, \mathcal{J})$ and $\mathsf{dist}^a_{\#}(\mathcal{I}, \mathcal{J})$, respectively.

Finally, one can also define distances at the level of the concept and role *symbols* in the signature Σ underlying the interpretations:

$$\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}) = \{S \in \Sigma \mid S^{\mathcal{I}} \neq S^{\mathcal{J}}\} \qquad \text{and} \qquad \mathsf{dist}^s_{\#}(\mathcal{I}, \mathcal{J}) = |\{S \in \Sigma \mid S^{\mathcal{I}} \neq S^{\mathcal{J}}\}|.$$

Summing up across the different possibilities, we have three dimensions with two values each: (1) *local* vs. *global* approach, (2) *atom*-based vs. *symbol*-based for defining distances, and (3) *set inclusion* vs. *cardinality* to compare symmetric differences. This gives eight evolution semantics, as shown in Figure 1, right. We denote each of the resulting eight semantics by using a combination of three symbols, indicating the choice in each dimension, e.g., $\mathbf{L}^a_{\#}$ denotes the local semantics where the distances are expressed in terms of cardinality of sets of atoms. We can then define $\mathsf{loc\_min}^y_x$ and $\mathbf{L}^y_x$-evolution (respectively, $\mathsf{glob\_min}^y_x$ and $\mathbf{G}^y_x$-evolution) as in Definition 3.1 (respectively, as in Definition 3.2) by using the specific distances determined by the values of $x \in \{\subseteq, \#\}$ and $y \in \{a, s\}$.

Recall that in Definitions 3.1 and 3.2, when we define the set of models $\mathsf{Mod}(\mathcal{K}')$, we indicate the specific evolution semantics S as a subscript of the evolution operator $\diamond$, i.e., as in $\mathcal{K} \diamond_S \mathcal{N}$. In terms of the introduced notation for the eight semantics, for, say, $\mathbf{G}^a_{\#}$ semantics, the set of models $\mathsf{Mod}(\mathcal{K}')$ should be denoted as $\mathcal{K} \diamond_{\mathbf{G}^a_{\#}} \mathcal{N}$. This notation is overloaded due to the composed nature the subscript $\mathbf{G}^a_{\#}$; to avoid this overload we will use the following convention instead: $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^a_{\#}$.

Closure Under Evolution and Approximation In the propositional case, each set of interpretations over finitely many symbols can be captured by a suitable formula, that is, a formula whose models are exactly those interpretations. In the case of DLs, this is no more necessarily the case, since, on the one hand, interpretations can be infinite, and, on the other hand, logics may miss some connectives, like disjunction or negation, that would be necessary to capture those interpretations.

Let $\mathcal{D}$ be a DL and $\mathcal{M}$ a set of models. We say that $\mathcal{M}$ is *axiomatizable* in $\mathcal{D}$ if there is a KB $\mathcal{K}$ such that $\mathsf{Mod}(\mathcal{K}) = \mathcal{M}$.

Definition 3.3 (Closure under Evolution)**.** Let S be an MBA. We say that a DL $\mathcal{D}$ is *closed under* S (or, evolution under S is *expressible* in $\mathcal{D}$) if for every $\mathcal{D}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ there is $\mathcal{K}' \in \mathcal{D}$ such that $\mathcal{K}'$ is an S-evolution for $\mathcal{E}$, i.e. $\mathsf{Mod}(\mathcal{K}') = \mathcal{K} \diamond_S \mathcal{N}$.

The notion of expressibility immediately suggests the following *expressibility problem.*

> **[EXPRESS]**: Does an S-evolution exist for a given $\mathcal{D}$-evolution setting and an MBA S?

It has been shown in [16, 18] that *DL-Lite* is not closed under any of the eight model-based semantics presented above. The observation underlying these results is that, on the one hand, the minimality of change principle intrinsically introduces implicit disjunction in the evolved KB. On the other hand, since *DL-Lite* is a slight extension of Horn logic [15], it does not allow one to express genuine disjunction (see Lemma 1 in [16] for details).

The negative answer to the **EXPRESS** problem for *DL-Lite* suggests the following *approximation problem*:

> **[APPROXIMATE]**: If S-evolution does not exists for a given $\mathcal{D}$-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ and an MBA S, is there a KB $\widetilde{\mathcal{K}} = \mathcal{T} \cup \widetilde{\mathcal{A}}$ which is a "good" approximation of $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$?

There are two commonly used notions of approximation for knowledge evolution: *sound* and *complete* approximations. In this paper we will address sound approximations only and leave the study of complete ones for the future work. We now define approximations formally.

Definition 3.4 (Sound Approximations)**.** Let $\mathcal{M}$ be a set of models and $\mathcal{D}$ a DL. We say that an $\mathcal{D}$ KB $\widetilde{\mathcal{K}}$ is a *sound $\mathcal{D}$-approximation* of $\mathcal{M}$ if $\mathcal{M} \subseteq \mathsf{Mod}(\widetilde{\mathcal{K}})$. Moreover, we say that a sound $\mathcal{D}$-approximation $\widetilde{\mathcal{K}}$ of $\mathcal{M}$ is *minimal* if for every sound $\mathcal{D}$-approximation $\widetilde{\mathcal{K}}_1$ of $\mathcal{M}$ it holds that $\mathsf{Mod}(\widetilde{\mathcal{K}}_1) \not\subset \mathsf{Mod}(\widetilde{\mathcal{K}})$.

3.2 Evolution of *DL-Lite*^{pr} KBs

In this section we consider how to capture evolution under $\mathbf{L}^a_\subseteq$ and $\mathbf{G}^s_\subseteq$ in *DL-Lite*^{pr}. Further, we will show that evolution under $\mathbf{L}^s_\subseteq$ in inexpressible in *DL-Lite*^{pr}; then, we will present an algorithm to compute minimal sound *DL-Lite*^{pr}-approximation of an evolution under this semantics. In Section 3.3 we will discuss how the results obtained in the current section can be applied to the rest of eight semantics.

3.2.1 Capturing $\mathbf{L}^a_\subseteq$-Evolution

Consider the algorithm AtAlg presented in Algorithm 1, which takes as input a *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ and returns the maximal subset of $\mathsf{cl}_\mathcal{T}(\mathcal{A})$ that is $\mathcal{T}$-satisfiable with $\mathcal{N}$. We illustrate how AtAlg works on the following example.

Example 3.3. Continuing with Examples 2.1, 2.2, and 2.3, we now illustrate how the algorithm AtAlg works on the following two evolution settings: $\mathcal{E}_1 = (\mathcal{T}_1 \cup \mathcal{A}_0, \mathcal{N}_1)$ and $\mathcal{E}_2 = (\mathcal{T}_1 \cup \mathcal{A}_0, \mathcal{N}_2)$.

Algorithm 1: $\mathsf{AtAlg}(\mathcal{E})$

INPUT : *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$
OUTPUT: The maximal set $\mathcal{A}' \subseteq \mathsf{cl}_\mathcal{T}(\mathcal{A})$ of ABox assertions that is $\mathcal{T}$-satisfiable with $\mathcal{N}$

$\mathcal{A}' := \emptyset;\quad X := \mathsf{cl}_\mathcal{T}(\mathcal{A});$
repeat
    choose some $\phi \in X;\quad X := X \setminus \{\phi\};$
    if $\{\phi\} \cup \mathcal{N} \not\models_\mathcal{T} \bot$ **then** $\mathcal{A}' := \mathcal{A}' \cup \{\phi\};$
until $X = \emptyset;$
return $\mathcal{A}'$

We remind the reader our notations:

$$\begin{array}{rcl}
\mathcal{T}_1 & = & \{\mathit{Card} \sqsubseteq \mathit{Priest}, \quad \mathit{Husb} \sqsubseteq \neg\mathit{Priest}\},\\
\mathcal{N}_1 & = & \{\mathit{Priest}(\mathit{john})\},\\
\mathcal{N}_2 & = & \{\mathit{Husb}(\mathit{pedro}),\ \mathit{Wife}(\mathit{tanya})\},\ \text{and}\\
\mathcal{A}_0 & = & \{\mathit{Priest}(\mathit{pedro}),\ \mathit{Priest}(\mathit{ivan}),\ \mathit{Husb}(\mathit{john}),\ \mathit{Wife}(\mathit{mary}),\ \mathit{Wife}(\mathit{chloe}),\\
& & \mathit{HasHusb}(\mathit{mary}, \mathit{john})\}.
\end{array}$$

Computation of both $\mathsf{AtAlg}(\mathcal{E}_1)$ and $\mathsf{AtAlg}(\mathcal{E}_2)$ rely on the computation of $\mathsf{cl}_{\mathcal{T}_1}(\mathcal{A}_0)$, which is equal to:

$$\mathsf{cl}_{\mathcal{T}_1}(\mathcal{A}_0) = \mathcal{A}_0 \cup \{\neg\mathit{Husb}(\mathit{pedro}),\ \neg\mathit{Husb}(\mathit{ivan}),\ \neg\mathit{Priest}(\mathit{john}),\ \neg\mathit{Card}(\mathit{john})\}.$$

Finally, by dropping from $\mathsf{cl}_{\mathcal{T}_1}(\mathcal{A}_0)$ atoms that conflict with $\mathcal{N}_1$ and $\mathcal{N}_2$ we obtain, respectively:

$$\mathsf{AtAlg}(\mathcal{E}_1) = \mathsf{cl}_\mathcal{T}(\mathcal{A}_0) \setminus \{\mathit{Husb}(\mathit{john}),\ \neg\mathit{Priest}(\mathit{john})\},$$

$$\mathsf{AtAlg}(\mathcal{E}_2) = \mathsf{cl}_\mathcal{T}(\mathcal{A}_0) \setminus \{\mathit{Priest}(\mathit{pedro}),\ \neg\mathit{Husb}(\mathit{pedro})\}. \qquad \blacksquare$$

We are going to prove that using AtAlg one can efficiently compute $\mathbf{L}^a_\subseteq$-evolutions in *DL-Lite*^{pr}. Before doing that, we will present the following definitions, auxiliary propositions, and lemma. Detailed proofs of proof-sketches can be found in the appendix of our online technical report [37].

Observe an important property of *DL-Lite*^{pr} KBs which shows that the source of inconsistency in these KBs comes from interaction between *unary* atoms only. As we will see in Section 3.4, inconsistency of KBs which are not in *DL-Lite*^{pr} can come from interaction involving binary atoms, which immediately leads to expressibility issues with evolution (see Section 3.4.1 for details).

Proposition 3.1. *For a DL-Lite*^{pr} *KB* $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ *and an assertion* g *of the form* $R(a, b)$ *or* $\exists R(a)$*, if* $\mathcal{A} \models_\mathcal{T} g$*, then also* $\mathsf{AtAlg}((\mathcal{K}, \mathcal{N})) \cup \mathcal{N} \models_\mathcal{T} g$*.*

Proof (Sketch). The proof is based on the facts that a *DL-Lite*^{pr} TBox $\mathcal{T}$ does not entail NIs of the form $B \sqsubseteq \neg\exists R$ for a basic concept B and a basic role R, and $\mathcal{N}$ does not contain negative MAs (due to Definition 2.2). Therefore, assertions of the form $R(a, b)$ or $\exists R(a)$ cannot $\mathcal{T}$-contradict $\mathcal{N}$. $\square$

The following proposition shows the cases when the union of models of $\mathcal{T}$ is also a model of $\mathcal{T}$.

Proposition 3.2. *Let* $\mathcal{T}$ *be a DL-Lite*$_{core}$ *TBox, and* $\mathcal{I}_1$, $\mathcal{I}_2$ *be models of* $\mathcal{T}$*. Then,* $\mathcal{I}_1 \cup \mathcal{I}_2 \models \mathcal{T}$ *if and only if for every* $f_1 \in \mathcal{I}_1$ *and* $f_2 \in \mathcal{I}_2$*, it holds that* $\{f_1, f_2\} \not\models_\mathcal{T} \bot$*.*

Proof (Sketch). The "only if" direction is trivial. Regarding the "if" direction, for the PIs of $\mathcal{T}$ it holds since they are satisfied in both $\mathcal{I}_1$ and $\mathcal{I}_2$; for the NIs of $\mathcal{T}$ it holds since the assumption of the "only if" direction prevents their violation. $\square$

We now define how to *uproot* an atom f from a model w.r.t. a TBox, i.e., how to delete the atoms g of the model from which f can be "deduced" using the TBox in the logic programming sense. We denote the set of these atoms g to be uprooted as $\mathsf{root}_{\mathcal{T}}(f)$. Formally:

Definition 3.5 ($\mathsf{root}_{\mathcal{T}}$). Let $\mathcal{T}$ be a *DL-Lite*$_{core}$ TBox and $V^n_{\mathcal{T}}$ a sequence $\langle f_1, \ldots, f_n, L\rangle$, where $f_1, \ldots, f_n$ are ground atoms and L is a ground literal, such that there is a sequence of TBox assertions $\alpha_1, \ldots, \alpha_n$ from $\mathsf{cl}(\mathcal{T})$, where each α_i is *not* of the form $\exists R \sqsubseteq \exists R$, $f_i \to f_{i+1}$ is an instantiation of the first-order interpretation of α_i[4] for $1 \leq i \leq n-1$, and $f_n \to L$ is an instantiation of the first-order interpretation of α_n. Note that α_n can be either a PI (if L is positive) or NI (if L is negative), while all the other α_is are PIs. Then,

$$\mathsf{root}_{\mathcal{T}}(C(a)) = \bigcup_{V^n_{\mathcal{T}}:\ n\in\mathbb{N},\ L=C(a)} V^n_{\mathcal{T}}, \qquad \mathsf{root}_{\mathcal{T}}(R(a,b)) = \bigcup_{\substack{V^n_{\mathcal{T}}:\ n\in\mathbb{N},\ L=R(a,d) \text{ or} \\ L=R(d,b) \text{ for some } d\in\Delta}} V^n_{\mathcal{T}}.$$

If $\mathcal{I}$ is an interpretation, then $\mathsf{root}^{I}_{\mathcal{T}}(C(a))$ denotes a restriction of $\mathsf{root}_{\mathcal{T}}(C(a))$ to $\mathcal{I}$, i.e., the subset of $\mathsf{root}_{\mathcal{T}}(C(a))$ where each $V^n_{\mathcal{T}}$ in the union satisfies $V_{\mathcal{T}} \subseteq \mathcal{I}$. Note that in the following whenever we write $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ for an MA g and a model $\mathcal{I}$, we always mean $\mathcal{I} \setminus \mathsf{root}^{I}_{\mathcal{T}}(g)$.

Example 3.4. Consider a TBox $\mathcal{T} = \{B \sqsubseteq \exists R^-,\ \exists R \sqsubseteq C\}$ and a model $\mathcal{I} = \{B(b),\ R(a,b),\ C(a)\}$. Note that $\mathcal{I} \models \mathcal{T}$. Let us see how an interpretation $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(C(a))$ (that is, $\mathcal{I} \setminus \mathsf{root}^{\mathcal{I}}_{\mathcal{T}}(C(a))$) looks like. Note that the restriction of $\mathsf{root}_{\mathcal{T}}(C(a))$ on $\mathcal{I}$ includes only one sequence of atoms of length three $V^2_{\mathcal{T}} = \langle f_1, f_2, L\rangle$ (the sequences of smaller lengths are "subsumed" by $V^2_{\mathcal{T}}$), where $f_1 = B(b)$, $f_2 = R(a,b)$, and $L = C(a)$; moreover, $B(b) \to R(a,b)$ instantiates $\alpha_1 = B \sqsubseteq \exists R^-$ and $R(a,b) \to C(a)$ instantiates $\alpha_2 = \exists R \sqsubseteq C$. Thus, $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(C(a)) = \emptyset$. ■

The following proposition shows that $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(C(a))$ is a model of $\mathcal{T}$ if $\mathcal{I}$ is a model of $\mathcal{T}$.

Proposition 3.3. *Let $\mathcal{T}$ be a DL-Lite$_{core}$ TBox and $\mathcal{I} \models \mathcal{T}$. Then, $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \models \mathcal{T}$ for every general MA g.*

Proof (Sketch). Let α be a TBox assertion such that $\mathcal{T} \models \alpha$. If α is an NI, then we conclude that $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \models \alpha$ since $\mathcal{I} \models \alpha$ and $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \subseteq \mathcal{I}$. If α is a PI of the form $B_1 \sqsubseteq B_2$, then assume that $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \not\models \alpha$. That is, there are atoms f_1 and f_2, that instantiate B_1 and B_2, respectively, such that $f_1 \in \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ and $f_2 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$. Here we conclude that $f_2 \in \mathsf{root}_{\mathcal{T}}(g)$; by the definition of $\mathsf{root}_{\mathcal{T}}$, we imply that $f_1 \in \mathsf{root}_{\mathcal{T}}(g)$, and therefore $f_1 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$. The latest statement contradicts the assumption above and concludes the proof. □

Now we present a lemma that will help us for a given model $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$ to construct a model $\mathcal{I} \models \mathcal{K}$ such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. This method of constructing $\mathcal{I}$ is the key for proving a number of results in this work. Intuitively, such $\mathcal{I}$ can be constructed in two steps: *(i)* drop from $\mathcal{J}$ all unary atomic MAs (i.e., unary atoms) that are *not* $\mathcal{T}$-satisfiable with $\mathcal{A}$, and then *(ii)* add unary atomic MAs that are $\mathcal{T}$-entailed from $\mathcal{A}$. The following sets will be used in the proof of this lemma below.

Definition 3.6. Let $\mathcal{T}$ be a TBox, $\mathcal{A}$ an ABox satisfiable with $\mathcal{T}$, then the *unary closure* of $\mathcal{A}$ w.r.t. $\mathcal{T}$ is

$$\mathsf{ucl}_{\mathcal{T}}(\mathcal{A}) = \{A(c) \mid A \text{ is an atomic concept, } c \text{ is a constant, and } \mathcal{A} \models_{\mathcal{T}} A(c)\}.$$

Moreover, let $\mathcal{J}$ be an interpretation, then the set of atoms of $\mathcal{J}$ which are in *conflict* with $\mathcal{A}$ w.r.t. $\mathcal{T}$ is

$$\mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A}) = \{A(c) \in \mathcal{J} \mid A \text{ is an atomic concept, } c \text{ is a constant, and } \mathcal{A} \cup \{A(c)\} \models_{\mathcal{T}} \bot\}.$$

[4] If α_i is of the form $A_1 \sqsubseteq A_2$ or $A_2 \sqsubseteq \exists R$, then the first-order interpretation of α_i is respectively the implication $A_1(x) \to A_2(x)$ or $A_2(x) \to R(x,y)$, where x and y are some variables, and this interpretation can be instantiated with, e.g., atoms $A_1(a)$, $A_2(a)$ and $R(a,b)$ as follows: $A_1(a) \to A_2(a)$ or $A_2(a) \to R(a,b)$.

Lemma 3.4. *Let $(\mathcal{K},\mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a DL-Lite^{pr}-evolution setting and let $\mathcal{J}$ be a model of $\mathcal{T} \cup \mathsf{AtAlg}((\mathcal{K},\mathcal{N})) \cup \mathcal{N}$. Then, the following interpretation is a model of $\mathcal{K}$:*

$$\mathcal{I} = (\mathcal{J} \setminus \mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})) \cup \mathsf{ucl}_{\mathcal{T}}(\mathcal{A}). \tag{5}$$

Moreover, $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I},\mathcal{T},\mathcal{N})$ and $\mathsf{dist}^a_{\subseteq}(\mathcal{I},\mathcal{J})$ is finite.

Proof. Finiteness of $\mathsf{dist}^a_{\subseteq}(\mathcal{I},\mathcal{J})$ follows from finiteness of $\mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})$ and $\mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$. Indeed, since $\mathcal{A} \models_{\mathcal{T}} A(c)$ for every $A(c) \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$, we conclude that $c \in \mathsf{adom}(\mathcal{K})$; since $\mathcal{A} \cup \{A(c)\} \models_{\mathcal{T}} \perp$ for every $A(c) \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})$, we conclude that $\mathcal{A} \models_{\mathcal{T}} \neg A(c)$, and again $c \in \mathsf{adom}(\mathcal{K})$. Due to the finiteness of $\Sigma(\mathcal{K})$, it holds that $|\mathsf{adom}(\mathcal{K})| \leq n$ and $|\{A \mid A \in \Sigma(\mathcal{K})\}| \leq m$ for some $n, m \in \mathbb{N}$. Hence, $|\mathsf{ucl}_{\mathcal{T}}(\mathcal{A})| \leq n \times m$ and $|\mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})| \leq n \times m$.

Now we prove that $\mathcal{I} \models \mathcal{K}$ by showing that $\mathcal{I} \models \mathcal{A}$ and $\mathcal{I} \models \mathcal{T}$. Afterwards, we will prove that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I},\mathcal{T},\mathcal{N})$. Let $\mathcal{A}' = \mathsf{AtAlg}((\mathcal{K},\mathcal{N})) \cup \mathcal{N}$.

$\mathcal{I} \models \mathcal{A}$: Let g be an MA such that $g \in \mathcal{A}$; we will show that $\mathcal{I} \models g$. We have three following cases:

(i) g is of the form $A(c)$; then $g \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$ and we conclude that $g \in \mathcal{I}$ by the definition of $\mathcal{I}$, so $\mathcal{I} \models g$.

(ii) g is of the form $R(a,b)$ or $\exists R(a)$; then, by Proposition 3.1, it holds that $g \in \mathcal{J}$. Since $\mathcal{I}$ and $\mathcal{J}$ differ on only unary atoms by the definition of $\mathcal{I}$, we conclude that $g \in \mathcal{I}$, so $\mathcal{I} \models g$.

(iii) g is of the form $\neg A(c)$; from $\neg A(c) \in \mathcal{A}$ and the definition of $\mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})$, we conclude that $A(c) \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})$. Thus, $A(c) \notin \mathcal{I}$ by the construction of $\mathcal{I}$ and therefore $\mathcal{I} \models g$.

From the three cases above we conclude that $\mathcal{I} \models \mathcal{A}$.

$\mathcal{I} \models \mathcal{T}$: We now show that $\mathcal{I} \models \mathcal{T}$ in two steps. First, observe that $\mathcal{J} \setminus \mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A}) \models \mathcal{T}$. Indeed, since $\mathcal{J} \models \mathcal{T}$, it is enough to show that for every $f \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})$, if $\{f'\} \models_{\mathcal{T}} f$ for some $f' \in \mathcal{J}$, then $f' \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})$. This is clearly the case because $\{f'\} \models_{\mathcal{T}} f$ and $\mathcal{A} \cup \{f\} \models_{\mathcal{T}} \perp$ imply $\mathcal{A} \cup \{f'\} \models_{\mathcal{T}} \perp$, and consequently $f' \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})$. Now we show that adding $\mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$ to $\mathcal{J} \setminus \mathsf{conf}_{\mathcal{T}}(\mathcal{J},\mathcal{A})$ does not violate $\mathcal{T}$. Indeed, let $f \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$, we need to show that for every assertion g such that $\{f\} \models_{\mathcal{T}} g$ it holds $\mathcal{I} \models g$. Clearly, g can only be of the form either $A(c)$ or $\exists R(a)$. If $g = A(c)$, then $g \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$ and obviously $\mathcal{I} \models g$. If $g = \exists R(a)$, then observe that $\{f\} \models_{\mathcal{T}} g$ implies $\mathcal{A} \models_{\mathcal{T}} g$; thus, due to Proposition 3.1, $\mathcal{A}' \models g$ and, as we showed above, $\mathcal{I} \models g$.

$\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I},\mathcal{T},\mathcal{N})$: By the definition of $\mathbf{L}^a_{\subseteq}$-evolution, we need to show that there is *no* $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ such that $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$. Assume there exists such $\mathcal{J}'$. Thus, there is an atom f such that $f \notin \mathcal{I} \ominus \mathcal{J}'$ while $f \in \mathcal{I} \ominus \mathcal{J}$. By the definition of $\mathcal{I}$, interpretations $\mathcal{I}$ and $\mathcal{J}$ differ only on atoms of the form $A(c)$; hence, f is of the form $A(c)$ (it cannot be of the form $R(a,b)$). We have two cases:

(i) $A(c) \in \mathcal{I}$, $A(c) \notin \mathcal{J}$, and $A(c) \in \mathcal{J}'$: By construction of $\mathcal{I}$, $A(c) \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$, while $A(c) \notin \mathcal{J}$ implies $\mathcal{A}(c) \notin \mathcal{A}'$. Thus, $\{A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \perp$. On the other hand, $A(c) \in \mathcal{J}'$ and $\mathcal{J}' \models \mathcal{N}$ imply that $\{A(c)\} \cup \mathcal{N} \not\models \perp$, which yields a contradiction.

(ii) $A(c) \notin \mathcal{I}$, $A(c) \in \mathcal{J}$, and $A(c) \notin \mathcal{J}'$: From $A(c) \notin \mathcal{J}'$ and $\mathcal{J}' \models \mathcal{N}$ we imply that $\mathcal{N} \not\models A(c)$. By the definition of $\mathcal{I}$, the assumptions $A(c) \notin \mathcal{I}$ and $A(c) \in \mathcal{J}$ imply that $\{A(c)\} \cup \mathcal{A} \models_{\mathcal{T}} \perp$, and therefore $\neg A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$. By the definition of $\mathcal{J}$, the assumption $A(c) \in \mathcal{J}$ implies $\neg A(c) \notin \mathsf{AtAlg}(\mathcal{K},\mathcal{N})$. From $\neg A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ and $\neg A(c) \notin \mathsf{AtAlg}(\mathcal{K},\mathcal{N})$ we conclude that $\{\neg A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \perp$, and, therefore, $\mathcal{N} \models A(c)$ holds, which yields a contradiction.

Thus, $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I},\mathcal{T},\mathcal{N})$ and we conclude the proof. □

Finally, consider the following definition of an auxiliary model $\mathcal{I}[g]$.

Definition 3.7. (Submodel $\mathcal{I}[g]$) If $\mathcal{I}$ is an interpretation and g is a positive MA, then $\mathcal{I}[g]$ is a minimal w.r.t. set inclusion submodel of $\mathcal{I}$ satisfying both g and $\mathcal{T}$.

We are now ready to state our result formally.

Theorem 3.5. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a DL-Lite^{pr}-evolution setting. Then,*

$$\mathcal{K}' = \mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N} \tag{6}$$

is a DL-Lite^{pr} KB and it is an $\mathbf{L}^a_{\subseteq}$-evolution for $\mathcal{E}$. Moreover, $\mathcal{K}'$ is computable in time polynomial in $|\mathcal{E}|$.

Proof. Polynomiality of AtAlg in $|\mathcal{K} \cup \mathcal{N}|$ follows from worst case quadratic size of $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ in $|\mathcal{K}|$ and polynomiality in $|\mathcal{K} \cup \mathcal{N}|$ of the test $\{\phi\} \cup \mathcal{N} \not\models_{\mathcal{T}} \bot$. Let $\mathcal{M}$ denote the set of models $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$. For simplicity we denote: $\mathcal{A}'' = \mathsf{AtAlg}(\mathcal{E})$ and $\mathcal{A}' = \mathcal{A}'' \cup \mathcal{N}$. Thus, $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$. We now prove that $\mathcal{K}'$ is an $\mathbf{L}^a_{\subseteq}$-evolution by showing two inclusions: $\mathcal{M} \subseteq \mathsf{Mod}(\mathcal{K}')$ and $\mathsf{Mod}(\mathcal{K}') \subseteq \mathcal{M}$.

$\mathcal{M} \subseteq \mathsf{Mod}(\mathcal{K}')$: Let $\mathcal{J} \in \mathcal{M}$, we will show $\mathcal{J} \in \mathsf{Mod}(\mathcal{K}')$, i.e., $\mathcal{J} \in \mathsf{Mod}(\mathcal{T})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{A}')$. By the definition of $\mathbf{L}^a_{\subseteq}$-evolution, $\mathcal{J} \in \mathcal{M}$ implies $\mathcal{J} \in \mathsf{Mod}(\mathcal{T})$.

Assume $\mathcal{J} \notin \mathsf{Mod}(\mathcal{A}')$. Since $\mathcal{J} \in \mathcal{M}$ we have $\mathcal{J} \models \mathcal{N}$. Since also $\mathcal{J} \notin \mathsf{Mod}(\mathcal{A}')$ we have $\mathcal{J} \not\models \mathcal{A}''$. Thus, there is an MA $g \in \mathcal{A}''$ such that $\mathcal{J} \not\models g$, where the assertion g can be either a positive MA or a negative one. From $\mathcal{J} \not\models g$, we will deduce a contradiction by showing $\mathcal{J} \notin \mathcal{M}$, that is, by showing that for every $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ there is $\mathcal{J}' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that

$$\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}. \tag{7}$$

(i) Assume g is a positive MA. Consider an arbitrary $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$. Clearly, $\mathcal{I} \models g$ (since $\mathcal{I} \models \mathcal{T} \cup \mathcal{A}$, $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}''$, and $g \in \mathcal{A}''$). Now let $\mathcal{J}' = \mathcal{J} \cup \mathcal{I}[g]$ (recall that $\mathcal{I}[g]$ is a minimal submodel of $\mathcal{I}$ satisfying both g and $\mathcal{T}$). Clearly, such $\mathcal{I}[g]$ exists while it may be not unique. If $\mathcal{I}[g]$ is not unique, then any such $\mathcal{I}[g]$ can be used in the construction of $\mathcal{J}'$.

Observe that $\mathcal{J}' \models \mathcal{N}$ and $\mathcal{J}' \models \mathcal{T}$. The former entailment holds since $\mathcal{J} \models \mathcal{N}$. Now we show the latter entailment. Assume $\mathcal{J}' \not\models \mathcal{T}$. Since $\mathcal{J} \models \mathcal{T}$, $\mathcal{I}[g] \models \mathcal{T}$, then, due to Proposition 3.2, there are $f_1 \in \mathcal{J}$ and $f_2 \in \mathcal{I}[g]$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. Note that, as a consequence of Proposition 3.1, both atoms f_1 and f_2 are unary. From $f_2 \in \mathcal{I}[g]$ and $\mathcal{I}[g] \subseteq \mathcal{I}$ we conclude that $f_2 \in \mathcal{I}$; combining $f_2 \in \mathcal{I}$ with $\mathcal{I} \models \mathcal{T}$ and $\{f_1, f_2\} \models_{\mathcal{T}} \bot$, we conclude that $f_1 \notin \mathcal{I}$.

Now we show that the conclusion $f_1 \notin \mathcal{I}$ leads to the contradiction with the fact that $\mathcal{J} \notin \mathcal{M}$, which will prove that $\mathcal{J}' \models \mathcal{T}$. To this effect we need to define another interpretation $\mathcal{J}_1$ in the following way: $\mathcal{J}_1 = \mathcal{J} \setminus \mathsf{root}_{\mathcal{T}}(f_1)$. We will show that $\mathcal{J}_1 \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}_1 \subsetneq \mathcal{I} \ominus \mathcal{J}$, thus $\mathcal{J} \notin \mathcal{M}$, which will give us a contradiction. The entailment $\mathcal{J}_1 \models \mathcal{T}$ holds by Proposition 3.3. To see that $\mathcal{J}_1 \models \mathcal{N}$ observe that $\mathcal{N} \not\models_{\mathcal{T}} f_1$. Indeed, if $\mathcal{N} \models_{\mathcal{T}} f_1$, then from $g \models_{\mathcal{T}} f_2$ and $\{f_1, f_2\} \models_{\mathcal{T}} \bot$ we can derive that $\mathcal{N} \cup \{g\} \models \bot$ which contradicts with the assumption that $g \in \mathcal{A}''$ (such g should have been dropped by AtAlg from $\mathcal{A}''$, see Line 4 of Algorithm 1). Using the definition of $\ominus$ and the facts that $\mathsf{root}^{\mathcal{J}}_{\mathcal{T}}(f_1) \cap \mathcal{I} = \emptyset$, and $(\mathcal{J} \setminus \mathsf{root}_{\mathcal{T}}(f_1)) \subseteq \mathcal{J}$ we now show that $\mathcal{I} \ominus \mathcal{J}_1 \subseteq \mathcal{I} \ominus \mathcal{J}$:

$$\begin{aligned}
\mathcal{I} \ominus \mathcal{J}_1 &= (\mathcal{I} \setminus \mathcal{J}_1) \cup (\mathcal{J}_1 \setminus \mathcal{I}) \\
&= \big(\mathcal{I} \setminus (\mathcal{J} \setminus \mathsf{root}_{\mathcal{T}}(f_1))\big) \cup \big((\mathcal{J} \setminus \mathsf{root}_{\mathcal{T}}(f_1)) \setminus \mathcal{I}\big) \\
&= (\mathcal{I} \setminus \mathcal{J}) \cup \big((\mathcal{J} \setminus \mathsf{root}_{\mathcal{T}}(f_1)) \setminus \mathcal{I}\big) \\
&\subseteq (\mathcal{I} \setminus \mathcal{J}) \cup (\mathcal{J} \setminus \mathcal{I}) = (\mathcal{I} \ominus \mathcal{J}).
\end{aligned}$$

Inequality $\mathcal{I} \ominus \mathcal{J}_1 \neq \mathcal{I} \ominus \mathcal{J}$ follows from the fact that $f_1 \notin \mathcal{I}$, $f_1 \notin \mathcal{J}_1$, and $f_1 \in \mathcal{J}$. This finishes the proof of $\mathcal{J}' \models \mathcal{T}$.

It remains to show that the Equation (7) holds for the constructed $\mathcal{J}'$. Since $\mathcal{I}[g] \subseteq \mathcal{I}$ one can apply the definition of $\ominus$ to conclude in the following way that $\mathcal{I} \ominus \mathcal{J}' = \mathcal{I} \ominus (\mathcal{J} \cup \mathcal{I}[g]) \subseteq \mathcal{I} \ominus \mathcal{J}$

indeed holds:

$$\begin{aligned}\mathcal{I} \ominus \mathcal{J}' &= (\mathcal{I} \setminus \mathcal{J}') \cup (\mathcal{J}' \setminus \mathcal{I}) \\ &= \big(\mathcal{I} \setminus (\mathcal{J} \cup \mathcal{I}[g])\big) \cup \big((\mathcal{J} \cup \mathcal{I}[g]) \setminus \mathcal{I}\big) \\ &= \big((\mathcal{I} \setminus \mathcal{J}) \cap (\mathcal{I} \setminus \mathcal{I}[g])\big) \cup (\mathcal{J} \setminus \mathcal{I}) \cup (\mathcal{I}[g] \setminus \mathcal{I}) \\ &\subseteq \big((\mathcal{I} \setminus \mathcal{J}) \cap \mathcal{I}\big) \cup (\mathcal{J} \setminus \mathcal{I}) \cup \emptyset \\ &= (\mathcal{I} \setminus \mathcal{J}) \cup (\mathcal{J} \setminus \mathcal{I}) = (\mathcal{I} \ominus \mathcal{J}).\end{aligned}$$

The inequality $\mathcal{I} \ominus \mathcal{J}' \neq \mathcal{I} \ominus \mathcal{J}$ follows from the fact that $g \in \mathcal{I}$, $g \in \mathcal{J}'$, and $g \notin \mathcal{J}$. Thus, $\mathcal{J} \notin \mathcal{M}$ and we obtain a contradiction.

(ii) Assume g is a negative MA, i.e., $g = \neg h$ for some positive MA h. Consider an arbitrary $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$. Clearly, $\mathcal{J} \models h$ and $\mathcal{I} \not\models h$ (since $\mathcal{I} \models \mathcal{T} \cup \mathcal{A}$, $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}''$, and $\neg h \in \mathcal{A}''$). Now let $\mathcal{J}' := \mathcal{J} \setminus \mathsf{root}_{\mathcal{T}}(h)$. Observe that $\mathcal{J}' \models \mathcal{N}$ and $\mathcal{J}' \models \mathcal{T}$. The former entailment holds since $\neg h \in \mathcal{A}''$ and consequently $\{\neg h\} \cup \mathcal{N} \not\models_{\mathcal{T}} \bot$; thus, $\mathsf{root}_{\mathcal{T}}(h) \cap \mathcal{N} = \emptyset$. The latter entailment holds due to Proposition 3.3.
Using the fact that $\mathcal{J}' \subseteq \mathcal{J}$, $\mathsf{root}_{\mathcal{T}}(h) \not\subseteq \mathcal{J}'$, $\mathsf{root}_{\mathcal{T}}(h) \not\subseteq \mathcal{I}$, and the definition of $\ominus$, one can show that $\mathcal{I} \ominus \mathcal{J}' \subseteq \mathcal{I} \ominus \mathcal{J}$. From the facts that $h \in \mathcal{J}$, $h \notin \mathcal{J}'$ and $h \notin \mathcal{I}$ we conclude that $\mathcal{I} \ominus \mathcal{J}' \neq \mathcal{I} \ominus \mathcal{J}$, and therefore $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$. We conclude that Equation (7) holds, $\mathcal{J} \notin \mathcal{M}$, and obtain a contradiction.

$\mathcal{M} \supseteq \mathsf{Mod}(\mathcal{K}')$: Let $\mathcal{J} \models \mathcal{K}'$. To prove that $\mathcal{J} \in \mathcal{M}$ we need to show that $\mathcal{J} \models \mathcal{T} \cup \mathcal{N}$ and there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. The former follows from the fact that $\mathcal{J} \models \mathcal{K}'$, while the latter from Lemma 3.4. Thus, $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ and therefore $\mathcal{M} \supseteq \mathsf{Mod}(\mathcal{K}')$ holds, which concludes the proof. □

We conclude this section with an example.

Example 3.5. In the notations of Example 3.3, due to Theorem 3.5 we have that $\mathbf{L}^a_{\subseteq}$-evolution for $(\mathcal{T} \cup \mathcal{A}_0, \mathcal{N}_1)$ is the following KB:

$$\mathcal{K}' = \mathcal{T} \cup \mathsf{AtAlg}(\mathcal{T} \cup \mathcal{A}_0, \mathcal{N}_1) \cup \{\mathit{Priest}(\mathit{john})\}.$$

This result is quite intuitive and expected: the new knowledge $\mathcal{N}_1$ asserts that *john* is a priest now, while the TBox $\mathcal{T}$ forbids to be a priest and a husband at once; thus, we have to drop from the old knowledge that *john* is a husband and that he is not a priest. Also note that $\mathcal{K}'$ contains $\neg \mathit{Card}(\mathit{john})$, that is, the fact that *john* became a priest did not make him a cardinal.

$\mathbf{L}^a_{\subseteq}$-evolution for $(\mathcal{T} \cup \mathcal{A}_0, \mathcal{N}_2)$ is the following KB:

$$\mathcal{K}' = \mathcal{T} \cup \mathsf{AtAlg}(\mathcal{T} \cup \mathcal{A}_0, \mathcal{N}_2) \cup \{\mathit{Husb}(\mathit{pedro}),\ \ \mathit{Wife}(\mathit{tanya})\}.$$

This result is again expected: *pedro* becomes a husband and we have to drop the old knowledge that he is a priest and not a husband. Moreover, *tanya* becomes a wife and, since this fact does not conflict with anything in the old knowledge, we just add it. ■

3.2.2 Capturing $\mathbf{G}^s_{\subseteq}$-Evolution

Consider the algorithm $\mathsf{GSymbAlg}$ in Algorithm 2. It takes a *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ as input. Then, it computes the set $\mathsf{AtAlg}(\mathcal{E})$ and for every atom ϕ in $\mathcal{N}$ it checks whether $\neg\phi \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ (Line 4). If it is the case, $\mathsf{GSymbAlg}$ deletes from $\mathsf{AtAlg}(\mathcal{E})$ all literals ϕ' that share the concept name with ϕ. Finally, $\mathsf{GSymbAlg}$ returns what remains from $\mathsf{AtAlg}(\mathcal{E})$. We will illustrate $\mathsf{GSymbAlg}$ with the following example.

Example 3.6. Consider a *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ with

$$\mathcal{T} = \{\mathit{Priest} \sqsubseteq \neg \mathit{Husb}\}, \quad \mathcal{A} = \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan})\}, \quad \text{and} \quad \mathcal{N} = \{\mathit{Husb}(\mathit{pedro})\}.$$

Algorithm 2: $\mathsf{GSymbAlg}(\mathcal{E})$

INPUT : *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$
OUTPUT : A set $\mathcal{A}' \subseteq \mathsf{cl}_\mathcal{T}(\mathcal{A}) \cup \mathsf{cl}_\mathcal{T}(\mathcal{N})$ of ABox assertions

1 $\mathcal{A}' := \emptyset$; $X := \mathsf{AtAlg}(\mathcal{E})$; $Y := \mathsf{cl}_\mathcal{T}(\mathcal{N})$;
2 **repeat**
3   **choose some** $\phi \in Y$; $Y := Y \setminus \{\phi\}$;
4   **if** $\neg\phi \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$ **then**
        $X := X \setminus \{\phi' \in \mathsf{cl}_\mathcal{T}(\mathcal{A}) \mid \phi$ and ϕ' have the same concept name$\}$
5 **until** $Y = \emptyset$;
6 $\mathcal{A}' := X \cup \mathcal{N}$;
7 **return** $\mathcal{A}'$;

Observe that $\mathsf{GSymbAlg}(\mathcal{E}) = \{Husb(pedro)\}$. Indeed, $Y = \mathsf{cl}_\mathcal{T}(\mathcal{N}) = \{Husb(pedro), \neg Priest(pedro)\}$; $\mathsf{cl}_\mathcal{T}(\mathcal{T} \cup \mathcal{A}) = \{Priest(pedro), Priest(ivan), \neg Husb(pedro)\}$, so $X = \mathsf{AtAlg}(\mathcal{E}) = \{Priest(ivan)\}$. Now, the assertion $\neg Priest(pedro)$ satisfies the condition of Line 4 of $\mathsf{GSymbAlg}$, and therefore the atom of X should be deleted, that is, $\mathsf{GSymbAlg}$ returns $\mathcal{A}' = \emptyset \cup \mathcal{N}$. ∎

We will show that $\mathsf{GSymbAlg}$ computes precisely $\mathbf{G}^s_\subseteq$-evolutions for *DL-Lite*^{pr}-evolution settings. Intuitively, $\mathsf{GSymbAlg}$ does so by tracing all assertions of the form $B(c)$ or $\neg B(c)$ entailed by $\mathcal{A}$ that should be deleted from (the $\mathcal{T}$-closure of) $\mathcal{A}$ due to $\mathcal{N}$. For such B's the change of interpretation is inevitable, i.e., if in some model $\mathcal{I}$ of $\mathcal{K}$ we had $b \in B^\mathcal{I}$, then in every $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^s_\subseteq$ we have $b \notin B^\mathcal{I}$. Since symbol-based semantics trace changes on symbols only, and the interpretation of the symbol B should be changed, one should drop from (the $\mathcal{T}$-closure of) $\mathcal{A}$ all the assertions over the symbol B, that is, of the form $B(d)$ and $\neg B(d)$ for some d. We will illustrate this phenomenon for $B = Priest$, $c = pedro$ and $d = ivan$ in the following example.

Example 3.7. Continuing with Example 3.6, observe that for every model $\mathcal{I} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{A})$, it holds $\mathcal{I} \models Priest(pedro)$, and for every model $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, it holds $\mathcal{J} \models \neg Priest(pedro)$. Hence, for every models $\mathcal{I} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{A})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, it holds that $\{Priest\} \leq \mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J})$, and therefore

$$\{Priest\} \leq \mathsf{dist}^s_\subseteq(\mathsf{Mod}(\mathcal{T} \cup \mathcal{A}), \mathcal{J}). \tag{8}$$

Consider the following models $\mathcal{J}_1, \mathcal{J}_2 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$:

$$\mathcal{J}_1 = \{Husb(pedro), Priest(ivan)\}, \qquad \mathcal{J}_2 = \{Husb(pedro)\}.$$

It is easy to see that for $\mathcal{I}_1 = \{Priest(pedro), Priest(ivan)\}$, we have: $\mathsf{dist}^s_\subseteq(\mathsf{Mod}(\mathcal{T} \cup \mathcal{A}), \mathcal{J}_i) \leq \mathsf{dist}^s_\subseteq(\mathcal{I}_1, \mathcal{J}_i) = \{Priest\}$ for $i = 1, 2$. Together with Equation (8), it gives that $\mathsf{dist}^s_\subseteq(\mathsf{Mod}(\mathcal{T} \cup \mathcal{A}), \mathcal{J}_i) = \{Priest\}$, so we conclude that $\mathcal{J}_i \in (\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $\mathcal{S} = \mathbf{G}^s_\subseteq$ and $i = 1, 2$. Observe that $\mathcal{J}_1 \models Priest(ivan)$ and $\mathcal{J}_2 \not\models Priest(ivan)$; thus, for $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$, which is $\mathbf{G}^s_\subseteq$-evolution for $\mathcal{E}$, it holds that $\mathcal{K}' \not\models Priest(ivan)$.

We emphasize that the behavior of $\mathbf{G}^s_\subseteq$-evolution is quire counterintuitive: as soon as we declare that a specific object is no longer in a concept, say A, (by asserting that it is in the complement to A, e.g., when we declared that *pedro* is no longer in *Priest* by asserting that he is in *Husb*), the old information about *all* the other objects in A should be erased (all old members of *Priest* should be erased). ∎

Before proceeding to a formal proof of correctness for $\mathsf{GSymbAlg}$, we present the following notations and a proposition. With $\mathcal{A} \parallel_\mathcal{T} \phi$ we denote the fact that neither $\mathcal{A} \models_\mathcal{T} \phi$ nor $\mathcal{A} \models_\mathcal{T} \neg\phi$ holds. The definition of $\mathcal{K} \parallel \phi$ is analogous. Observe that for every KB $\mathcal{K}$ and atom $A(c)$, there are three possible relations between them: $\mathcal{K} \models A(c)$, or $\mathcal{K} \models \neg A(c)$, or $\mathcal{K} \parallel A(c)$. For a given

(T1)	$\mathcal{N} \models_\mathcal{T} A(c)$	and	$\mathcal{K} \models A(c)$;	*(T6)*	$\mathcal{N} \models_\mathcal{T} \neg A(c)$	and	$\mathcal{K} \parallel A(c)$;
(T2)	$\mathcal{N} \models_\mathcal{T} \neg A(c)$	and	$\mathcal{K} \models \neg A(c)$;	*(T7)*	$\mathcal{N} \parallel_\mathcal{T} A(c)$	and	$\mathcal{K} \parallel A(c)$;
(T3)	$\mathcal{N} \models_\mathcal{T} A(c)$	and	$\mathcal{K} \models \neg A(c)$;	*(T8)*	$\mathcal{N} \parallel_\mathcal{T} A(c)$	and	$\mathcal{K} \models A(c)$;
(T4)	$\mathcal{N} \models_\mathcal{T} \neg A(c)$	and	$\mathcal{K} \models A(c)$;	*(T9)*	$\mathcal{N} \parallel_\mathcal{T} A(c)$	and	$\mathcal{K} \models \neg A(c)$.
(T5)	$\mathcal{N} \models_\mathcal{T} A(c)$	and	$\mathcal{K} \parallel A(c)$;				

Figure 2 – Classification of atoms $A(c)$ w.r.t. $\mathcal{K}$ and $\mathcal{N}$

$\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, $\mathcal{N}$, and atom $A(c)$, these three relations give nine combinations, which are presented in Figure 2. We refer to each such combination as *type of* $A(c)$ (w.r.t. $\mathcal{K}$ and $\mathcal{N}$) and consequently there are nine types: *(T1)-(T9)*.

We recall that $\mathcal{J}_0[A(c)]$ is a minimal submodel of $\mathcal{J}_0$ containing $A(c)$ and satisfying $\mathcal{T}$.

Proposition 3.6. *Let $\mathcal{T}$ be a DL-Lite^{pr} TBox, $\mathcal{I}$ and $\mathcal{J}$ models of $\mathcal{T}$, and $A(c)$ an atom. Then, the interpretation $(\mathcal{I} \setminus \mathsf{root}_\mathcal{T}(\neg A(c))) \cup \mathcal{J}[A(c)]$ is a model of $\mathcal{T}$.*

We proceed to correctness of $\mathsf{GSymbAlg}$.

Theorem 3.7. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a DL-Lite^{pr}-evolution setting. Then,*

$$\mathcal{K}' = \mathcal{T} \cup \mathsf{GSymbAlg}(\mathcal{E}) \tag{9}$$

is a DL-Lite^{pr} KB and $\mathbf{G}^s_\subseteq$-evolution for $\mathcal{E}$. Moreover, $\mathsf{GSymbAlg}(\mathcal{E})$ is computable in time polynomial in $|\mathcal{E}|$.

Proof. Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ and $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$. The fact that $\mathcal{K}'$ is a *DL-Lite^{pr}* KB follows from the fact that $\mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$ is in *DL-Lite^{pr}* (see Theorem 3.5) and $\mathcal{K}' \subseteq \mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$. Polynomiality of $\mathsf{GSymbAlg}$ follows from polynomiality of AtAlg, the fact that $|Y|$ is worst case quadratic in $|\mathcal{N} \cup \mathcal{T}|$, and that the test $\neg\phi \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$ is polynomial in $|\mathcal{K} \cup \mathcal{N}|$. Let $\mathcal{M} = \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^s_\subseteq$ and $\mathcal{M}' = \mathsf{Mod}(\mathcal{K}')$. We now show that $\mathcal{M} \subseteq \mathcal{M}'$ and $\mathcal{M}' \subseteq \mathcal{M}$.

$\mathcal{M} \subseteq \mathcal{M}'$: Consider a model $\mathcal{J}_0 \in \mathcal{M}$. We show that $\mathcal{J}_0 \in \mathcal{M}'$. By the definition of $\mathbf{G}^s_\subseteq$, we have $\mathcal{J}_0 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ and there exists a model $\mathcal{I}_0 \in \mathsf{Mod}(\mathcal{K})$ such that for every pair of models $\mathcal{J}_1 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ and $\mathcal{I}_1 \in \mathsf{Mod}(\mathcal{K})$ the following equation *does not* hold.

$$\mathsf{dist}^s_\subseteq(\mathcal{I}_1, \mathcal{J}_1) \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}_0, \mathcal{J}_0), \tag{10}$$

Assume that $\mathcal{J}_0 \notin \mathcal{M}'$. We now exhibit a pair of appropriate $\mathcal{I}_1$ and $\mathcal{J}_1$ that satisfies Equation (10), thus, obtaining a contradiction. Since $\mathcal{J}_0 \notin \mathcal{M}'$ and $\mathcal{J}_0 \models \mathcal{T} \cup \mathcal{N}$, so, by Line 6 of $\mathsf{GSymbAlg}$ (see Algorithm 2), $\mathcal{J}_0 \not\models X$. Hence, there exists a literal $L(c) \in X$ such that $\mathcal{J}_0 \not\models L(c)$. Proposition 3.1 and the fact that $\neg\phi \notin \mathsf{cl}_\mathcal{T}(\mathcal{A})$, where ϕ is of the form $R(a, b)$ or $\exists R(a)$, for a *DL-Lite^{pr}* KB imply that $L(c)$ is of the form $A(c)$ or $\neg A(c)$. Moreover, from $\mathcal{I}_0 \models \mathcal{K}$ we conclude that $\mathcal{I}_0 \models X$ and consequently $\mathcal{I}_0 \models L(c)$. Therefore, $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$.

Now we construct $\mathcal{I}_1$ and $\mathcal{J}_1$ from $\mathcal{I}_0$ and $\mathcal{J}_0$, respectively, in a way that they agree on the interpretation of A. The construction of these models depends on the type of $A(d) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I} \ominus \mathcal{J})$ for $d \in \Delta$ w.r.t. $\mathcal{K}$, $\mathcal{N}$ (Figure 2). Observe that $A(d)$ *cannot* be of type *(T1)-(T4)*. Indeed, if $A(d)$ is of type *(T1)* or *(T2)*, then $A(d) \notin \mathcal{I}_0 \ominus \mathcal{J}_0$. If $A(d)$ is of type *(T3)* or *(T4)*, then $A(d) \in \mathcal{I} \ominus \mathcal{J}$. Both cases contradict $A(d) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I} \ominus \mathcal{J})$.

We construct $\mathcal{I}_1$ from $\mathcal{I}_0$ using atoms $A(d)$ of type *(T5)-(T7)*, and then $\mathcal{J}_1$ from $\mathcal{J}_0$ using atoms $A(d)$ of type *(T7)-(T9)*. The interpretation $\mathcal{I}_1$ is defined as follows:

$$\mathcal{I}_1 := \bigcup_{\substack{A(d) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\ \mathcal{J}_0 \models A(d) \\ A(d) \text{ of type } (T5) \text{ or } (T7)}} \Big((\mathcal{I}_0 \setminus \mathsf{root}_\mathcal{T}(\neg A(d))) \cup \mathcal{J}_0[A(d)] \Big) \setminus \bigcup_{\substack{A(d) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\ \mathcal{J}_0 \models \neg A(d) \\ A(d) \text{ of type } (T6)}} \mathsf{root}_\mathcal{T}(A(d)). \tag{11}$$

Observe that $\mathcal{I}_1 \models \mathcal{K}$. Indeed, due to Proposition 3.3 and Proposition 3.6 we have that $\mathcal{I}_1 \models \mathcal{T}$. To see that $\mathcal{I}_1 \models \mathcal{A}$, recall that $A(d)$ is of type *(T5)-(T7)* and therefore $\mathcal{K} \parallel A(d)$. Moreover, one can show that due to the fact that $\mathcal{K}$ is in *DL-Lite*^{pr} and $\mathcal{J}_0 \models \neg A(d)$, each subtracted set $\mathsf{root}_\mathcal{T}(A(d))$ contains only unary atoms of the form $A'(d)$. Combining these two observation we conclude that $\mathcal{K} \parallel A'(d)$ and therefore $A'(d)$ $notin\mathcal{A}$. Thus, $\mathcal{I}_1 \models \mathcal{A}$.

The interpretation $\mathcal{J}_1$ is defined as follows:

$$\mathcal{J}_1 := \bigcup_{\substack{A(d) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\ \mathcal{I}_0 \models A(d) \\ A(d) \text{ of type } (T8) \text{ or } (T7)}} \Big((\mathcal{J}_0 \setminus \mathsf{root}_\mathcal{T}(\neg A(d))) \cup \mathcal{I}_0[A(d)] \Big) \setminus \bigcup_{\substack{A(d) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\ \mathcal{I}_0 \models \neg A(d) \\ A(d) \text{ of type } (T9)}} \mathsf{root}_\mathcal{T}(A(d)). \tag{12}$$

One can show that $\mathcal{J}_1 \models \mathcal{T} \cup \mathcal{N}$ analogously to the proof of $\mathcal{I}_1 \models \mathcal{T} \cup \mathcal{A}$ above. Observe that by construction of $\mathcal{I}_1$ and $\mathcal{J}_1$, we have $A^{\mathcal{I}_1} = A^{\mathcal{J}_1}$ and $\mathsf{dist}^s_\subseteq(\mathcal{I}_1, \mathcal{J}_1) \subseteq \mathsf{dist}^s_\subseteq(\mathcal{I}_0, \mathcal{J}_0)$. Finally, the former equality gives that $A \notin \mathsf{dist}^s_\subseteq(\mathcal{I}_1, \mathcal{J}_1)$, which together with $A \in \mathsf{dist}^s_\subseteq(\mathcal{I}_0, \mathcal{J}_0)$ implies Equation (10) and concludes the proof.

$\mathcal{M}' \subseteq \mathcal{M}$: Let $\mathcal{J}_0 \in \mathcal{M}' = \mathsf{Mod}(\mathcal{T} \cup \mathcal{A}')$ where $\mathcal{A}' = \mathsf{GSymbAlg}(\mathcal{E})$, and assume $\mathcal{J}_0 \notin \mathcal{M}$, that is: *(i)* $\mathcal{J}_0 \notin \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, or *(ii)* for every $\mathcal{I} \models \mathcal{K}$ there is a pair of models $\mathcal{I}' \models \mathcal{K}$ and $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ s.t. $\mathsf{dist}^s_\subseteq(\mathcal{I}', \mathcal{J}') \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J}_0)$. Case (i) is impossible since $\mathcal{N} \subseteq \mathcal{A}'$. If Case (ii) holds, then consider a model $\mathcal{I}_0$ as in Equation (5). By Lemma 3.4 we have $\mathcal{I}_0 \models \mathcal{K}$. By our assumption, $\mathsf{dist}^s_\subseteq(\mathcal{I}', \mathcal{J}') \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}_0, \mathcal{J}_0)$ holds for some $\mathcal{I}'$ and $\mathcal{J}'$. Due to Proposition 3.1, $\mathcal{I}_0$ and $\mathcal{J}_0$ coincide on how they interpret roles. Thus, there is a concept A such that $A^{\mathcal{I}'} = A^{\mathcal{J}'}$ while $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$, and consequently there is an atom $A(c) \in \mathcal{I}_0 \ominus \mathcal{J}_0$. Note that, by the construction of $\mathcal{I}_0$, it holds that $A(c) \in \mathsf{ucl}_\mathcal{T}(\mathcal{A})$ or $A(c) \in \mathsf{conf}_\mathcal{T}(\mathcal{J}_0, \mathcal{A})$. We have two cases:

(i) $A(c) \in \mathcal{I}_0 \setminus \mathcal{J}_0$: From $A(c) \in \mathcal{I}_0$ we conclude that $A(c) \in \mathsf{ucl}_\mathcal{T}(\mathcal{A}) \subseteq \mathsf{cl}_\mathcal{T}(\mathcal{A})$. From $A(c) \notin \mathcal{J}_0$ and $\mathcal{J}_0 \models \mathcal{K}'$, we conclude that $A(c) \notin X$ (see Line 6 in Algorithm 2). From these two statements, $A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$ and $A(c) \notin X$, we imply that for some constant $b \in \mathsf{adom}(\mathcal{T} \cup \mathcal{A})$ (including c) one of the two cases holds: (a) $\neg A(b) \in \mathsf{cl}_\mathcal{T}(\mathcal{N})$ and $A(b) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$, or (b) $A(b) \in \mathsf{cl}_\mathcal{T}(\mathcal{N})$ and $\neg A(b) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$. Either case together with $\mathcal{J}' \models \mathsf{cl}_\mathcal{T}(\mathcal{N})$ and $\mathcal{I}' \models \mathsf{cl}_\mathcal{T}(\mathcal{A})$ implies $A^{\mathcal{I}'} \neq A^{\mathcal{J}'}$ and yield a contradiction with $A^{\mathcal{I}'} = A^{\mathcal{J}'}$.

(ii) $A(c) \in \mathcal{J}_0 \setminus \mathcal{I}_0$: In this case $A(c) \in \mathsf{conf}_\mathcal{T}(\mathcal{J}_0, \mathcal{A})$ (see Equation (5)) which means that $\{A(c)\} \cup \mathsf{cl}_\mathcal{T}(\mathcal{A}) \models_\mathcal{T} \bot$ and consequently $\neg A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$. From $\mathcal{J}_0 \models A(c)$ we conclude that $\neg A(c) \notin X$. Finally, from the two statements $\neg A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$ and $\neg A(c) \notin X$ we conclude that $A^{\mathcal{I}'} \neq A^{\mathcal{J}'}$ using the same argument as for $A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$ and $A(c) \notin X$ of Case (i) above.

Thus, $\mathcal{J}_0 \in \mathcal{M}$, which concludes the proof. □

3.2.3 Approximation of $\mathbf{L}^s_\subseteq$-Evolution

We start with an observation that $\mathbf{L}^s_\subseteq$ is not expressible in *DL-Lite*^{pr}, because capturing $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^s_\subseteq$ requires disjunction, which is not available in *DL-Lite*. Formally:

Theorem 3.8. *DL-Lite*^{pr} *is not closed under* $\mathbf{L}^s_\subseteq$ *semantics.*

Proof. Let $S = \mathbf{L}^s_\subseteq$. Consider the KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, where: $\mathcal{T} = \{A \sqsubseteq B\}$, $\mathcal{A} = \{B(c)\}$, and $\mathcal{N} = \{B(d)\}$. We now show that *(i)* every $\mathcal{J} \models \mathcal{K} \diamond_S \mathcal{N}$ satisfies $A(d) \rightarrow B(c)$, and *(ii)* there are models $\mathcal{J}_0, \mathcal{J}_1 \in \mathcal{K} \diamond_S \mathcal{N}$ such that $\mathcal{J}_1 \not\models \neg A(c)$ and $\mathcal{J}_2 \not\models B(c)$.Due to Lemma 1 in [16], if these two conditions hold, then $\mathcal{K} \diamond_S \mathcal{N}$ is inexpressible in *DL-Lite*, and hence in *DL-Lite*^{pr}.

To see that Condition (i) holds, assume there is a model $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ such that $\mathcal{J} \not\models A(d) \rightarrow B(c)$, i.e., $A(d) \in \mathcal{J}$ but $B(c) \notin \mathcal{J}$. By the definition of $\mathbf{L}^s_\subseteq$, there is $\mathcal{I} \models \mathcal{K}$ such that: for every $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ it *does not* hold that $\mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J}') \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J})$. Since $\mathcal{I} \models \mathcal{K}$ and $B(c) \in \mathcal{I}$ we have that $B \in \mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J})$. There are two cases:

- If $A(d) \in \mathcal{I}$, then also $B(d) \in \mathcal{I}$; thus, $\mathcal{I} \models \mathcal{T} \cup \mathcal{N}$ and, by taking $\mathcal{J}' = \mathcal{I}$, one obtains $\mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J}') \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J})$ which yields a contradiction.

Algorithm 3: $\mathsf{LSymbAlg}(\mathcal{E})$

INPUT : *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$
OUTPUT: A set $\mathcal{A}'' \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \cup \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ of ABox assertions

```
1 A'' := ∅;  X := AtAlg(T ∪ A, N);  Y := cl_T(N);
2 repeat
3     choose some φ ∈ Y;  Y := Y \ {φ};
4     if φ ∉ X then X := X \ {φ' ∈ cl_T(A) | φ and φ' have the same concept name}
5 until Y = ∅;
6 A'' := X ∪ N;
7 return A'';
```

- If $A(d) \notin \mathcal{I}$, then $\{A, B\} \in \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})$. Now consider an interpretation $\mathcal{J}' = \mathcal{I} \cup \{B(d)\}$, which is clearly a model of $\mathcal{T} \cup \mathcal{N}$. If $B(d) \in \mathcal{I}$ then $\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}') = \emptyset$, otherwise $\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}') = \{B\}$. In either case $\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})$, which yields a contradiction.

Thus, every $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ satisfies $A(d) \to B(c)$.

To see that Condition (ii) holds, consider the following interpretations $\mathcal{J}_1$ and $\mathcal{J}_2$:

$$\mathcal{J}_1 = \{A(d),\ B(d),\ B(c)\}, \qquad \mathcal{J}_2 = \{B(d)\}.$$

It is easy to see that *(a)* $\mathcal{J}_i \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ for $i = 1, 2$, *(b)* $\mathcal{J}_1 \not\models \neg A(d)$, and *(c)* $\mathcal{J}_2 \not\models B(c)$. It remains to show that $\mathcal{J}_i \in \mathcal{K} \diamond_S \mathcal{N}$, so the conditions of Lemma 1 in [16] will be satisfied, and inexpressibility of $\mathbf{L}^s_{\subseteq}$ in *DL-Lite*^{pr} will be proved. Let us show this.

- $\mathcal{J}_1 \in \mathcal{K} \diamond_S \mathcal{N}$: Note that $\mathcal{J}_1 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{A})$, so we conclude that $\mathsf{dist}^s_{\subseteq}(\mathcal{J}_1, \mathcal{J}_1) = \emptyset$. Thus, $\mathcal{J}_1 \in \mathsf{loc\_min}^s_{\subseteq}(\mathcal{J}_1, \mathcal{T}, \mathcal{N})$ and therefore $\mathcal{J}_1 \in \mathcal{K} \diamond_S \mathcal{N}$.
- $\mathcal{J}_2 \in \mathcal{K} \diamond_S \mathcal{N}$: Consider an interpretation $\mathcal{I}_2 = \{B(c)\}$, which is clearly a model of $\mathcal{T} \cup \mathcal{A}$. Note that $\mathsf{dist}^s_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2) = \{B\}$. Then, for every model $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, it holds that $\{B\} \subseteq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_2, \mathcal{J})$ since $\mathcal{I}_2 \not\models B(d)$ and $\mathcal{J} \models B(d)$. Thus, we conclude that it *does not* hold $\mathsf{dist}^s_{\subseteq}(\mathcal{I}_2, \mathcal{J}) \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2)$, that is, $\mathcal{J}_2 \in \mathsf{loc\_min}^s_{\subseteq}(\mathcal{I}_2, \mathcal{T}, \mathcal{N})$ and therefore $\mathcal{J}_2 \in \mathcal{K} \diamond_S \mathcal{N}$.

Thus, Conditions (i) and (ii) hold and we conclude the proof. □

The theorem above suggests to look for *DL-Lite*^{pr}-approximations of $\mathbf{L}^s_{\subseteq}$-evolution. We now show that the algorithm $\mathsf{LSymbAlg}$ in Algorithm 3 can be used for this purpose. Note that $\mathsf{LSymbAlg}$ differs from $\mathsf{GSymbAlg}$ in Line 4 only, i.e., $\mathsf{LSymbAlg}$ in Line 4 performs a test different from the one of $\mathsf{GSymbAlg}$. Intuitively, for an assertion ϕ of the form $A(c)$, $\mathsf{LSymbAlg}$ checks whether $A(c)$ is in $\mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ but not in X, and, if it is the case, then $\mathsf{LSymbAlg}$ deletes all the assertions over the concept A from $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$. Note that the test of $\mathsf{LSymbAlg}$ is weaker than the one of $\mathsf{GSymbAlg}$ since it is easier to get changes in the interpretation of A by choosing a model of $\mathcal{K}$ that does not include $A(c)$. We illustrate $\mathsf{LSymbAlg}$ on the following example.

Example 3.8. Consider a *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ with

$$\mathcal{T} = \emptyset, \quad \mathcal{A} = \{\mathit{Wife}(\mathit{mary}), \mathit{Wife}(\mathit{chloe})\}, \quad \text{and} \quad \mathcal{N} = \{\mathit{Wife}(\mathit{tanya})\}.$$

Observe that $\mathsf{GSymbAlg}(\mathcal{E}) = \{\mathit{Wife}(\mathit{tanya})\}$. Indeed, $Y = \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) = \mathcal{N}$, $\mathsf{cl}_{\mathcal{T}}(\mathcal{T} \cup \mathcal{A}) = \mathcal{A}$, and $X = \mathsf{AtAlg}(\mathcal{E}) = \mathcal{A}$. Now, the assertion $\mathit{Wife}(\mathit{tanya})$ satisfies the condition of Line 4 of $\mathsf{GSymbAlg}$, and therefore both atoms of X should be deleted, that is, $\mathsf{GSymbAlg}$ returns $\mathcal{A}' = \emptyset \cup \mathcal{N}$.

Observe that the KB $\mathcal{K}'' = \mathcal{T} \cup \mathsf{LSymbAlg}(\mathcal{E})$ that approximates $\mathbf{L}^s_{\subseteq}$-evolution for $\mathcal{E}$ is even less intuitive than $\mathbf{G}^s_{\subseteq}$-evolution $\mathcal{K}'$ for $\mathcal{E}$ from Example 3.6: $\mathbf{L}^s_{\subseteq}$-evolution erases all the old ABox information about a concept, say B, (e.g., such a B is *Wife* in our case) as soon as we just add any new object in B that does not even conflict with anything in the old ABox (in our case we added $\mathit{Wife}(\mathit{tanya})$ and had to erase the information about the other two wives, $\mathit{Wife}(\mathit{mary})$ and $\mathit{Wife}(\mathit{chloe})$, from the old knowledge). ■

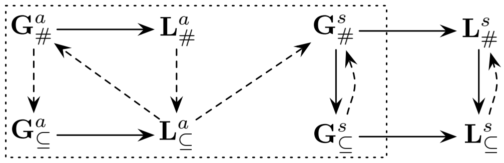

Figure 3 – Subsumptions for evolution semantics. The arrows stand for the subsumption $\preccurlyeq_{\mathsf{sem}}$: " $\longrightarrow$ ": for any DL (Theorem 3.10). " $\dashrightarrow$ ": for *DL-Lite*^{pr} (Theorems 3.12, 3.14, 3.16). The dashed frame surrounds those semantics under which *DL-Lite*^{pr} is closed.

Theorem 3.9. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a DL-Lite^{pr}-evolution setting. Then,*

$$\mathcal{K}'' = \mathcal{T} \cup \mathsf{LSymbAlg}(\mathcal{E}) \tag{13}$$

is a minimal sound DL-Lite^{pr}-approximation of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^s_{\subseteq}$. Moreover, $\mathsf{LSymbAlg}(\mathcal{E})$ *is computable in time polynomial in $|\mathcal{E}|$.*

Proof. The fact that $\mathcal{K}''$ is a *DL-Lite*^{pr} KB follows from the fact that $\mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$ is in *DL-Lite*^{pr} (see Theorem 3.5) and $\mathcal{K}'' \subseteq \mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$. Polynomiality of $\mathsf{LSymbAlg}$ can be shown analogously to polynomiality of $\mathsf{GSymbAlg}$ (see Theorem 3.7). Let $S = \mathbf{L}^s_{\subseteq}$. The fact that $\mathcal{K}''$ is a sound approximation of $\mathcal{K} \diamond_S \mathcal{N}$, i.e., $\mathcal{K} \diamond_S \mathcal{N} \subseteq \mathsf{Mod}(\mathcal{K}'')$, can also be shown analogously to the soundness $\mathcal{M} \subseteq \mathcal{M}'$ in Theorem 3.7.

Let $\mathcal{A}'' = \mathsf{LSymbAlg}(\mathcal{E})$. Suppose that $\mathcal{K}''$ is not a minimal sound approximation, which means we may add an assertion $A(c)$ to A'', where $A(c)$ is such that $\mathcal{K}'' \not\models A(c)$. That is, $\mathcal{K}''_1 = \mathcal{T} \cup \mathcal{A}'' \cup \{A(c)\}$ is another sound approximation. Consider a canonical model $\mathcal{J}''$ of $\mathcal{K}''$. Using a similar argument as in the proof of the completeness $\mathcal{M}' \subseteq \mathcal{M}$ in Theorem 3.7, one can show that $\mathcal{J}'' \in \mathcal{K} \diamond_S \mathcal{N}$. Clearly, $A(c) \notin \mathcal{J}''$, thus $\mathcal{J}'' \not\models \mathcal{K}''_1$ which contradicts the fact that $\mathcal{K}''_1$ is a sound approximation. □

Summary of Section 3.2 $\mathbf{L}^a_{\subseteq}$ and $\mathbf{G}^s_{\subseteq}$-evolutions for *DL-Lite*^{pr}-evolution settings can be computed in polynomial time; $\mathbf{L}^s_{\subseteq}$-evolution for a *DL-Lite*^{pr}-evolution setting in general does not exist, but one can find a minimal sound *DL-Lite*^{pr}-approximation of it in polynomial time.

3.3 Relationships between Model-Based Semantics

In this section we define a framework for comparing different model-based evolution semantics and apply it to the eight semantics that have been presented in Section 3.1.

Definition 3.8 (Subsumption on Evolution Semantics)**.** Let S_1 and S_2 be two evolution semantics, $\mathcal{D}$ a DL. Then, S_1 *is subsumed by* S_2 w.r.t. $\mathcal{D}$, denoted $(S_1 \preccurlyeq_{\mathsf{sem}} S_2)(\mathcal{D})$, or just $S_1 \preccurlyeq_{\mathsf{sem}} S_2$ when $\mathcal{D}$ is clear from the context, if $\mathcal{K} \diamond_{S_1} \mathcal{N} \subseteq \mathcal{K} \diamond_{S_2} \mathcal{N}$ for every $\mathcal{D}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$. Finally, S_1 and S_2 are *equivalent* w.r.t. $\mathcal{D}$, denoted $(S_1 \equiv_{\mathsf{sem}} S_2)(\mathcal{D})$, if $(S_1 \preccurlyeq_{\mathsf{sem}} S_2)(\mathcal{D})$ and $(S_2 \preccurlyeq_{\mathsf{sem}} S_1)(\mathcal{D})$. ■

The following theorem shows the subsumption relation between different semantics, independently of the chosen DL. We depict these relations in Figure 3 using solid arrows. Note that Figure 3 is complete in the following sense: there is a solid oriented path (a sequence of solid arrows) from a semantics S_1 to a semantics S_2 *if* $S_1 \preccurlyeq_{\mathsf{sem}} S_2(\mathcal{D})$ for every DL $\mathcal{D}$.

Theorem 3.10. *Let $x \in \{\subseteq, \#\}$ and $y \in \{a, s\}$ and. Then, for any DL it holds that*

$$\mathbf{G}^y_x \preccurlyeq_{\mathsf{sem}} \mathbf{L}^y_x, \qquad \mathbf{L}^s_{\#} \preccurlyeq_{\mathsf{sem}} \mathbf{L}^s_{\subseteq}, \qquad \textit{and} \qquad \mathbf{G}^s_{\#} \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_{\subseteq}.$$

Proof. We will consider all the three cases one by one.

$\mathbf{G}^y_x \preccurlyeq_{\mathsf{sem}} \mathbf{L}^y_x$: Let dist^y_x be a distance function and $(\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ a $\mathcal{D}$-evolution setting, where $\mathcal{D}$ is a DL. Let $\mathcal{M}_{\mathcal{G}} = \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{G}^y_x$ and $\mathcal{M}_{\mathcal{L}} = \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{L}^y_x$ be sets of models defined using respectively global (see Definition 3.2) and local (see Definition 3.1) semantics based on dist_x^y. Let $\mathcal{J}' \in \mathcal{M}_\mathcal{G}$, then there is $\mathcal{I}' \models \mathcal{K}$ such that for every $\mathcal{I}'' \models \mathcal{K}$ and $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ it *does not* hold that

$$\mathsf{dist}_x^y(\mathcal{I}'', \mathcal{J}'') \lneq \mathsf{dist}_x^y(\mathcal{I}', \mathcal{J}').$$

In particular, when $\mathcal{I}'' = \mathcal{I}'$, there is no $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ such that $\mathsf{dist}_x^y(\mathcal{I}', \mathcal{J}'') \lneq \mathsf{dist}_x^y(\mathcal{I}', \mathcal{J}')$, which yields that $\mathcal{J}' \in \mathsf{loc\_min}_x^y(\mathcal{I}', \mathcal{T}, \mathcal{N})$, and hence $\mathcal{J}' \in \mathcal{M}_\mathcal{L}$, which concludes the proof.

$\mathbf{L}_\#^s \preccurlyeq_{\mathsf{sem}} \mathbf{L}_\subseteq^s$: Consider $\mathcal{M}_\# = \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{L}_\#^s$, which is based on the distance $\mathsf{dist}_\#^s$, and $\mathcal{M}_\subseteq = \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{L}_\subseteq^s$, which is based on $\mathsf{dist}_\subseteq^s$. We now are interested in establishing whether $\mathcal{M}_\# \subseteq \mathcal{M}_\subseteq$ holds. Assume $\mathcal{J}' \in \mathcal{M}_\#$ and $\mathcal{J}' \notin \mathcal{M}_\subseteq$. Then, from the former assumption we conclude existence of $\mathcal{I}' \models \mathcal{K}$ such that $\mathcal{J}' \in \mathsf{loc\_min}_\#^s(\mathcal{I}', \mathcal{T}, \mathcal{N})$. From the latter assumption, $\mathcal{J}' \notin \mathcal{M}_\subseteq$, we conclude existence of a model $\mathcal{J}''$ such that $\mathsf{dist}_\subseteq^s(\mathcal{I}', \mathcal{J}'') \subsetneq \mathsf{dist}_\subseteq^s(\mathcal{I}', \mathcal{J}')$. Since the signature of $\mathcal{K} \cup \mathcal{N}$ is finite, the distance $\mathsf{dist}_\subseteq^s$ between every two models over this signature is also finite. Thus, we obtain that $\mathsf{dist}_\#(\mathcal{I}', \mathcal{J}'') \lneq \mathsf{dist}_\#(\mathcal{I}', \mathcal{J}')$, which contradicts to $\mathcal{J}' \in \mathcal{M}_\#$ and concludes the proof.

$\mathbf{G}_\#^s \preccurlyeq_{\mathsf{sem}} \mathbf{G}_\subseteq^s$: analogous to $\mathbf{L}_\#^s \preccurlyeq_{\mathsf{sem}} \mathbf{L}_\subseteq^s$. □

3.3.1 Relationships between Atom-Based Semantics in *DL-Lite*^{pr}

The next theorem shows that in *DL-Lite*^{pr} all four atom-based semantics coincide. Before proceeding to the formal statement and proof of this result, we present a technical proposition which is an analog of Proposition 3.1 for $\mathbf{L}_\#^a$ evolution semantics, i.e., Proposition 3.1 shows which MAs of the original KB are preserved by evolution under $\mathbf{L}_\subseteq^a$, while the following proposition – under $\mathbf{L}_\#^a$.

Proposition 3.11. *Let* $(\mathcal{K}, \mathcal{N})$*, where* $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$*, be a DL-Lite*^{pr}*-evolution setting, and* $\mathcal{M} = \mathcal{K} \diamond_S \mathcal{N}$ *with* $S = \mathbf{L}_\#^a$*. Let* g *be an MA of the form:* $R(a, b)$ *or* $\exists R(a)$*. If* $\mathcal{A} \models_\mathcal{T} g$*, then for every* $\mathcal{J} \in \mathcal{M}$ *it holds* $\mathcal{I} \models g$*.*

Theorem 3.12. *For DL-Lite*^{pr} *we have that* $\mathbf{L}_\#^a \equiv_{\mathsf{sem}} \mathbf{L}_\subseteq^a \equiv_{\mathsf{sem}} \mathbf{G}_\#^a \equiv_{\mathsf{sem}} \mathbf{G}_\subseteq^a$*.*

Proof. Due to Theorem 3.10 and transitivity of the relation $\preccurlyeq_{\mathsf{sem}}$, it suffices to show only three relations to conclude the proof: $\mathbf{L}_\#^a \preccurlyeq_{\mathsf{sem}} \mathbf{L}_\subseteq^a$, $\mathbf{L}_\subseteq^a \preccurlyeq_{\mathsf{sem}} \mathbf{G}_\#^a$, and $\mathbf{G}_\#^a \preccurlyeq_{\mathsf{sem}} \mathbf{G}_\subseteq^a$. Consider a *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$.

$\mathbf{L}_\#^a \preccurlyeq_{\mathsf{sem}} \mathbf{L}_\subseteq^a$: Consider a model $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}_\#^a$. Then, there is a model $\mathcal{I} \models \mathcal{K}$ such that $\mathcal{J} \in \mathsf{loc\_min}_\#^a(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Now consider a model $\mathcal{I}'$ built as in Equation (5); then, we have that $\mathsf{dist}_\#^a(\mathcal{I}, \mathcal{J}) \leq \mathsf{dist}_\#^a(\mathcal{I}', \mathcal{J})$. The latter distance is finite by Lemma 3.4, so is the former distance, and therefore $\mathbf{L}_\#^a \preccurlyeq_{\mathsf{sem}} \mathbf{L}_\subseteq^a$ (it can be shown in a similar way as $\mathbf{L}_\#^s \preccurlyeq_{\mathsf{sem}} \mathbf{L}_\subseteq^s$ has been shown in Theorem 3.10). Finally, the only point that remains to be shown to conclude the proof is that $\mathcal{I}' \models \mathcal{K}$. This can be shown similarly to the prove of Lemma 3.4 using Proposition 3.11 instead of Proposition 3.1.

$\mathbf{L}_\subseteq^a \preccurlyeq_{\mathsf{sem}} \mathbf{G}_\#^a$. Let $\mathcal{M}_L = \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}_\subseteq^a$ and $\mathcal{M}_G = \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}_\#^a$. Consider a model $\mathcal{J} \in \mathcal{M}_L$. Now we will show that $\mathcal{J} \in \mathcal{M}_G$, that is, that there is a model $\mathcal{I} \models \mathcal{K}$ such that for every $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I}' \models \mathcal{K}$ it *does not* hold that $|\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I} \ominus \mathcal{J}|$. Consider $\mathcal{I}$ as in Equation (5). Due to Lemma 3.4, $\mathcal{I} \models \mathcal{K}$. Assume there are $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I}' \models \mathcal{T} \cup \mathcal{A}$ such that $|\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I} \ominus \mathcal{J}|$. Since the set $\mathcal{I} \ominus \mathcal{J}$ is at most countable, $\mathcal{I}' \ominus \mathcal{J}'$ is finite, so there exists an atom $A(c) \in (\mathcal{I} \ominus \mathcal{J}) \setminus (\mathcal{I}' \ominus \mathcal{J}')$. We have two cases:

(i) $A(c) \in \mathcal{I} \setminus \mathcal{J}$: By the definition of $\mathcal{I}$, this condition implies that $A(c) \in \mathsf{ucl}_\mathcal{T}(\mathcal{A})$. Observe that $\mathsf{ucl}_\mathcal{T}(\mathcal{A}) \subseteq \mathsf{cl}_\mathcal{T}(\mathcal{A})$, and consequently $A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$. Due to Theorem 3.5, the inclusion $\mathcal{J} \in \mathcal{M}_L$ implies that $\mathcal{J}$ is a model of $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$, where $\mathcal{A}' = \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$. Due to $A(c) \notin \mathcal{J}$, we conclude that $A(c) \notin \mathsf{AtAlg}(\mathcal{E})$. From the last condition and $A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$, we obtain $\{A(c)\} \cup \mathcal{N} \models_\mathcal{T} \bot$ (this follows from the definition of AtAlg). This entailment together with $\mathcal{J}' \models \mathcal{N}$ implies that $A(c) \notin \mathcal{J}'$. From $A(c) \notin \mathcal{J}'$ and $A(c) \notin \mathcal{I}' \ominus \mathcal{J}'$ we get

$A(c) \notin \mathcal{I}'$. Finally, since $A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$ and $\mathcal{I}' \models \mathsf{cl}_\mathcal{T}(\mathcal{A})$, we have that $A(c) \in \mathcal{I}'$, which yields a contradiction.

(ii) $A(c) \in \mathcal{J} \setminus \mathcal{I}$: By the definition of $\mathcal{I}$, this condition implies $A(c) \in \mathsf{conf}_\mathcal{T}(\mathcal{J}, \mathcal{A})$ and also $\{A(c)\} \cup \mathsf{cl}_\mathcal{T}(\mathcal{A}) \models_\mathcal{T} \bot$, which is equivalent to $\neg A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$. Thus, $A(c) \notin \mathcal{I}'$ (otherwise $\mathcal{I}'$ would not be a model of $\mathcal{K}$). Recall that $A(c) \notin \mathcal{I}' \ominus \mathcal{J}'$, so $A(c) \notin \mathcal{J}'$. We obtain that $\mathcal{J}' \not\models A(c)$. Since $A(c) \in \mathcal{J}$, we have $\neg A(c) \notin \mathcal{A}'$, that is, $\{\neg A(c)\} \cup \mathcal{N} \models_\mathcal{T} \bot$. Now, combining the last entailment with $\mathcal{J}' \models \mathcal{N}$, we conclude $\mathcal{J}' \not\models \neg A(c)$, which contradicts $\mathcal{J}' \not\models A(c)$.

Thus, $\mathcal{J} \in \mathcal{M}_G$ and consequently $\mathbf{L}^a_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{G}^a_\#$.

$\mathbf{G}^a_\# \preccurlyeq_{\mathsf{sem}} \mathbf{G}^a_\subseteq$. Let $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$. Assume $\mathcal{J} \in \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{G}^a_\#$, but $\mathcal{J} \notin \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{G}^a_\subseteq$. The former assumption implies that there is a model $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ that is $\mathbf{G}^a_\#$-minimally distant from $\mathcal{J}$. The latter assumption implies the existence of models $\mathcal{I}' \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J}' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ s.t.

$$\mathcal{I}' \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J} \tag{14}$$

Due to the finite model property of *DL-Lite*^{pr}, it holds that $|\mathcal{I} \ominus \mathcal{J}|$ is finite.[5] This inequality together with Equation (14) yields that $|\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I} \ominus \mathcal{J}|$, which contradicts the fact that $\mathcal{I}$ is $\mathbf{G}^a_\#$-minimally distant from $\mathcal{J}$ and therefore concludes the proof. □

From Theorems 3.5 and 3.12 we conclude the following:

Corollary 3.13. *For DL-Lite*^{pr}*-evolution settings* $\mathcal{E}$*, the knowledge base* $\mathcal{K}'$ *of Equation* (6) *is an* S*-evolution, where* $S \in \{\mathbf{L}^a_\#, \mathbf{G}^a_\#, \mathbf{G}^a_\subseteq\}$.

3.3.2 Relationships between Symbol-Based Semantics in *DL-Lite*^{pr}

For symbol-based semantics, the local semantics based on cardinality and on set inclusion coincide, as well the global ones, while local semantics are not subsumed by the global ones.

Theorem 3.14. *For DL-Lite*^{pr} *we have that:* $\mathbf{L}^s_\subseteq \equiv_{\mathsf{sem}} \mathbf{L}^s_\#$*, and* $\mathbf{G}^s_\subseteq \equiv_{\mathsf{sem}} \mathbf{G}^s_\#$*, while* $\mathbf{L}^s_\subseteq \not\preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\#$.

Proof. We will consider all the three cases one by one.

$\mathbf{G}^s_\# \equiv_{\mathsf{sem}} \mathbf{G}^s_\subseteq$: Let $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a *DL-Lite*^{pr}-evolution setting. Due to Theorem 3.10, it suffices to show $\mathbf{G}^s_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\#$. Let $\mathcal{M}_\subseteq = \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^s_\subseteq$ and $\mathcal{M}_\# = \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^s_\#$. Consider a model $\mathcal{J}_0 \in \mathcal{M}_\subseteq$, we show that $\mathcal{J}_0 \in \mathcal{M}_\#$. By the definition of $\mathbf{G}^s_\subseteq$ semantics, there is a model $\mathcal{I}_0 \in \mathsf{Mod}(\mathcal{K})$, such that for every pair of models $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ it *does not* hold that $\mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J}) \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}_0, \mathcal{J}_0)$. Suppose that $\mathcal{J}_0 \notin \mathcal{M}_\#$, that is, for each model $\mathcal{I}' \in \mathsf{Mod}(\mathcal{K})$ there are models $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $|\mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J})| \lneq |\mathsf{dist}^s_\subseteq(\mathcal{I}', \mathcal{J}_0)|$. In particular, it holds when $\mathcal{I}' = \mathcal{I}_0$. This implies that there is an element in the signature of $\mathcal{K} \cup \mathcal{N}$ with the same interpretation in $\mathcal{I}$ and $\mathcal{J}$, and different interpretations in $\mathcal{I}_0$ and $\mathcal{J}_0$. If this element is a concept A, then $A^\mathcal{I} = A^\mathcal{J}$ and $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$ (the case when this element is a role symbol is analogous). Thus, there is an atom $A(c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I} \ominus \mathcal{J})$ for some $c \in \Delta$. We now exhibit models $\mathcal{I}_1 \models \mathcal{K}$ and $\mathcal{J}_1 \models \mathcal{T} \cup \mathcal{N}$ s.t.

$$\mathsf{dist}^s_\subseteq(\mathcal{I}_1, \mathcal{J}_1) \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}_0, \mathcal{J}_0), \tag{15}$$

which contradicts the assumption $\mathcal{J}_0 \in \mathcal{M}_\subseteq$. Now we construct $\mathcal{I}_1$ and $\mathcal{J}_1$ as in Equations (11) and (12), respectively. The proof that Equation (15) holds for these $\mathcal{I}_1$ and $\mathcal{J}_1$ is similar to the the proof that Equation (10) holds for $\mathcal{I}_1$ and $\mathcal{J}_1$ in Theorem 3.7. Thus, $\mathcal{J}_0 \in \mathcal{M}_\#$.

$\mathbf{L}^s_\# \equiv_{\mathsf{sem}} \mathbf{L}^s_\subseteq$: Due to Theorem 3.10, it suffices to show $\mathbf{L}^s_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{L}^s_\#$. This can be done similarly to the case of $\mathbf{G}^s_\# \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\subseteq$, by proving $\mathsf{dist}^s_\subseteq(\mathcal{I}_0, \mathcal{J}_1) \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}_0, \mathcal{J}_0)$ with $\mathcal{J}_1$ for $A(c)$ of types *(T7)-(T9)*.

[5] See a detailed proof of this statement in Proposition A.2 in the appendix.

$\mathbf{L}^s_\subseteq \not\preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\#$: Consider the following evolution setting $(\mathcal{K},\mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$: $\mathcal{T} = \{B \sqsubseteq \neg C\}$, $\mathcal{A} = \{A(c), B(a), B(d)\}$, and $\mathcal{N} = \{A(e), C(d)\}$. Consider the following model of $\mathcal{T} \cup \mathcal{N}$: $\mathcal{J} = \{A(e), C(d)\}$. To conclude the proof, observe that for $S_1 = \mathbf{L}^s_\subseteq$ and $S_2 = \mathbf{G}^s_\#$ it holds that $\mathcal{J} \in \mathcal{K} \diamond_{S_1} \mathcal{N}$ and $\mathcal{J} \notin \mathcal{K} \diamond_{S_2} \mathcal{N}$. □

From Theorems 3.7, 3.9 and 3.14 we conclude the following:

Corollary 3.15. *DL-Lite*^{pr} *is not closed under* $\mathbf{L}^s_\#$*-evolution and for DL-Lite*^{pr}*-evolution settings* $\mathcal{E} = (\mathcal{K},\mathcal{N})$

(i) the KB $\mathcal{K}'$ *of Equation* (9) *is an* $\mathbf{G}^s_\#$*-evolution;*

(ii) the KB $\mathcal{K}''$ *of Equation* (13) *is minimal sound DL-Lite*^{pr}*-approximation of* $\mathcal{K} \diamond_S \mathcal{N}$ *with* $S = \mathbf{L}^s_\#$.

3.3.3 Symbol vs. Atom-Based Semantics in *DL-Lite*^{pr}

Finally, we show that all atom-based semantics are subsumed by the global symbol-based semantics, while the contrary does not hold. Due to Theorems 3.10, 3.12, and 3.14, this statement follows from $\mathbf{L}^a_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\#$ and $\mathbf{G}^s_\# \not\preccurlyeq_{\mathsf{sem}} \mathbf{L}^a_\subseteq$. Thus, for *DL-Lite*^{pr} we essentially have three different evolution semantics: atom-based, local symbol-based, and global symbol-based.

Theorem 3.16. *For DL-Lite*^{pr} *we have that:* $\mathbf{L}^a_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\#$*, while* $\mathbf{G}^s_\# \not\preccurlyeq_{\mathsf{sem}} \mathbf{L}^a_\subseteq$.

Proof. We will consider the two cases one by one.

$\mathbf{L}^a_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\#$: Let $(\mathcal{K},\mathcal{N})$ be a *DL-Lite*^{pr}-evolution setting. Consider a model $\mathcal{J} \in \mathcal{K} \diamond_{S_1} \mathcal{N}$, where $S_1 = \mathbf{L}^a_\subseteq$. We will show that the following inclusion holds: $\mathcal{J} \in \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 \in \mathbf{G}^s_\#$. Due to Theorem 3.5, $\mathcal{J}$ is a model of $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$ as in Equation (6). Let $\mathcal{I}$ be an interpretation built as in Equation (5). Due to Lemma 3.4, we obtain $\mathcal{I} \models \mathcal{K}$. Suppose that $\mathcal{J} \notin \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{G}^s_\#$, that is, there exists a model $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ such that $|\mathsf{dist}^s_\subseteq(\mathsf{Mod}(\mathcal{K}), \mathcal{J}')| \lneq |\mathsf{dist}^s_\subseteq(\mathsf{Mod}(\mathcal{K}), \mathcal{J})|$. By the definition of the distance between a set of interpretations and an interpretation, there exists a model $\mathcal{I}' \models \mathcal{K}$ such that $|\mathsf{dist}^s_\subseteq(\mathcal{I}', \mathcal{J}')| \lneq |\mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J})|$. This implies that there is an element in the signature of $\mathcal{K} \cup \mathcal{N}$ with the same interpretation in $\mathcal{I}'$ and $\mathcal{J}'$, while it is interpreted differently in $\mathcal{I}$ and $\mathcal{J}$. Note that there is no role $P \in \Sigma(\mathcal{K} \cup \mathcal{N})$ such that $P^\mathcal{I} \neq P^\mathcal{J}$ due to the construction of $\mathcal{I}$. We consider the case when this element is a concept A, i.e., $A^\mathcal{I} \neq A^\mathcal{J}$ and $A^{\mathcal{I}'} = A^{\mathcal{J}'}$. From $A^\mathcal{I} \neq A^\mathcal{J}$ and Equation (5) we imply that there is an atom $A(c)$ that is either in $\mathsf{conf}_\mathcal{T}(\mathcal{J}, \mathcal{A})$ or in $\mathsf{ucl}_\mathcal{T}(A)$.

- If $A(c) \in \mathsf{conf}_\mathcal{T}(\mathcal{J}, \mathcal{A})$, then $\neg A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$, and, since $\mathcal{J} \not\models \neg A(c)$, the literal $\neg A(c)$ was deleted from $\mathsf{cl}_\mathcal{T}(\mathcal{A})$ while building $\mathcal{A}'$ (see Algorithm 1), i.e., $\{\neg A(c)\} \cup \mathcal{N} \models_\mathcal{T} \bot$. From this entailment and $\mathcal{J}' \models \mathcal{N}$ we conclude that $\mathcal{J}' \not\models \neg A(c)$ and consequently $\mathcal{I}' \not\models \neg A(c)$ (since $A^{\mathcal{I}_1} = A^{\mathcal{J}_1}$). We obtain a contradiction with $\mathcal{I}_1 \models \mathsf{cl}_\mathcal{T}(\mathcal{A})$.

- If $A(c) \in \mathsf{ucl}_\mathcal{T}(A)$, then $A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$ and $\mathcal{I}' \models A(c)$. Due to $A^{\mathcal{I}'} = A^{\mathcal{J}'}$, we have that $\mathcal{J}' \models A(c)$. On the other hand, since $\mathcal{I} \models \mathsf{ucl}_\mathcal{T}(\mathcal{A})$ and $A^\mathcal{I} \neq A^\mathcal{J}$, we conclude that $\mathcal{J} \not\models A(c)$. Thus, $A(c)$ was deleted from $\mathsf{cl}(\mathcal{A})$ while building $\mathcal{A}'$ (see Algorithm 1) and therefore $\{A(c)\} \cup \mathcal{N} \models_\mathcal{T} \bot$. Recall that $\mathcal{J}' \models \mathcal{N}$, thus, $\mathcal{J}' \not\models A(c)$ and we obtain a contradiction.

$\mathbf{G}^s_\# \not\preccurlyeq_{\mathsf{sem}} \mathbf{L}^a_\subseteq$: Consider the evolution setting $(\mathcal{K},\mathcal{N})$ as in the proof of the case $\mathbf{L}^s_\subseteq \not\preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\#$ in Theorem 3.14 and take the following model of $\mathcal{T} \cup \mathcal{N}$: $\mathcal{J} = \{A(c), A(e), C(d)\}$. To conclude the proof observe that, for $S_1 = \mathbf{G}^s_\#$ and $S_2 = \mathbf{L}^a_\subseteq$ it holds that $\mathcal{J} \in \mathcal{K} \diamond_{S_1} \mathcal{N}$ and $\mathcal{J} \notin \mathcal{K} \diamond_{S_2} \mathcal{N}$. □

Summary on Section 3.3 and *DL-Lite*^{pr} Atom-based approaches (which all coincide) can be captured using a polynomial-time algorithm AtAlg. Moreover, the evolution results produced under these MBAs are intuitive and expected. Symbol-based approaches on contrary produce quite unexpected and counterintuitive results (these semantics delete too much data).

Two global symbol-based approaches coincide and can be captured using a polynomial-time algorithm $\mathsf{GSymbAlg}$. Two local symbol-based approaches also coincide, cannot be captured in $DL\text{-}Lite^{pr}$, and can be approximated using a polynomial-time algorithm $\mathsf{LSymbAlg}$. Based on these results we conclude that using atom-based approaches for applications seems to be more practical. In Figure 3, using dashed arrows, we illustrate all the subsumptions between semantics discovered in this section. Note that Figure 3 is complete for $DL\text{-}Lite^{pr}$ in the following sense: there is an oriented path with solid or dashed arrows (a sequence of such arrows) between any two semantics S_1 and S_2 *if and only if* $(S_1 \preccurlyeq_{\mathsf{sem}} S_2)(DL\text{-}Lite^{pr})$. Moreover, in Figure 3 we framed in a dashed rectangle the six out of eight MBAs under which $DL\text{-}Lite^{pr}$ is closed.

3.4 Understanding $\mathbf{L}^a_\subseteq$-Evolution of $DL\text{-}Lite_{core}$ KBs

In the previous section we showed that atom-based MBAs behave better than symbol-based ones for $DL\text{-}Lite^{pr}$-evolution. This suggests to investigate atom-based MBAs for the entire $DL\text{-}Lite_{core}$. Here we focus on one of these four semantics, namely $\mathbf{L}^a_\subseteq$. The remaining three atom-based MBAs are subjects of future work.

As a further motivation for the study of $\mathbf{L}^a_\subseteq$, note that $\mathbf{L}^a_\subseteq$ is essentially the same as so-called *Winslett's semantics* [50] (WS), that was widely studied in the literature [41, 22]. Liu, Lutz, Milicic, and Wolter studied WS for expressive DLs [41]. Most of the DLs they considered are not closed under WS. Poggi, Lembo, De Giacomo, Lenzerini, and Rosati studied WS in the similar setting as the one adopted in this paper. They called it instance-level update for *DL-Lite* [22] and proposed an algorithm to compute the result of updates. However, the algorithm turned out to have technical issues, and it was shown that it was neither sound nor complete [16]. Note that the extension of this algorithm that approximates ABox updates in fragments of *DL-Lite* [22] inherits these technical issues. Actually, such an ABox update algorithm cannot exist since it was shown that *DL-Lite* is not closed under ABox evolution under $\mathbf{L}^a_\subseteq$ [18].

The remaining part of the section is organized as follows. In Section 3.4.1 we explain *why* $DL\text{-}Lite_{core}$ is not closed under $\mathbf{L}^a_\subseteq$ and show which combination of $DL\text{-}Lite_{core}$ formulas is responsible for inexpressibility. In Section 3.4.2 we introduce so-called prototypes that give a characterization of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ and are further used to approximate $\mathbf{L}^a_\subseteq$-evolution. In Section 3.4.3 we present a procedure BP that constructs prototypes for $DL\text{-}Lite_{core}$ evolution settings and Section 3.4.4 we show correctness of BP.

3.4.1 Understanding Inexpressibility of $\mathbf{L}^a_\subseteq$-Evolution in $DL\text{-}Lite_{core}$

Using the following example, we illustrate why $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ is not expressible in $DL\text{-}Lite_{core}$.

Example 3.9. Consider a $DL\text{-}Lite_{core}$ KB $\mathcal{K}_2 = \mathcal{T}_2 \cup \mathcal{A}_2$, new information $\mathcal{N}_1$, and $\mathcal{I} \models \mathcal{K}_2$:

$$\begin{aligned}
\mathcal{T}_2 &= \{\mathit{Wife} \sqsubseteq \exists \mathit{HasHusb}, \exists \mathit{HasHusb}^- \sqsubseteq \neg \mathit{Priest}\};\\
\mathcal{A}_2 &= \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan}), \exists \mathit{HasHusb}^-(\mathit{john})\};\\
\mathcal{N}_1 &= \{\mathit{Priest}(\mathit{john})\};
\end{aligned}$$

$$\mathcal{I}: \quad \mathit{Wife}^{\mathcal{I}} = \{\mathit{girl}\}, \quad \mathit{Priest}^{\mathcal{I}} = \{\mathit{pedro}, \mathit{ivan}\}, \quad \mathit{HasHusb}^{\mathcal{I}} = \{(\mathit{girl}, \mathit{john})\},$$

where $\mathit{girl} \in \Delta \setminus \mathsf{adom}(\mathcal{K}_2)$ is an element of the domain. The following models belong to $\mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}_2, \mathcal{N}_1)$ and consequently to $\mathcal{K}_2 \diamond_S \mathcal{N}_1$ with $S = \mathbf{L}^a_\subseteq$:

$$\begin{array}{llll}
\mathcal{J}_0: & \mathit{Wife}^{\mathcal{J}_0} = \emptyset, & \mathit{Priest}^{\mathcal{J}_0} = \{\mathit{john}, \mathit{pedro}, \mathit{ivan}\}, & \mathit{HasHusb}^{\mathcal{J}_0} = \emptyset,\\
\mathcal{J}_1: & \mathit{Wife}^{\mathcal{J}_1} = \{\mathit{girl}\}, & \mathit{Priest}^{\mathcal{J}_1} = \{\mathit{john}, \mathit{ivan}\}, & \mathit{HasHusb}^{\mathcal{J}_1} = \{(\mathit{girl}, \mathit{pedro})\},\\
\mathcal{J}_2: & \mathit{Wife}^{\mathcal{J}_2} = \{\mathit{girl}\}, & \mathit{Priest}^{\mathcal{J}_2} = \{\mathit{john}, \mathit{pedro}\}, & \mathit{HasHusb}^{\mathcal{J}_2} = \{(\mathit{girl}, \mathit{ivan})\},\\
\mathcal{J}_3: & \mathit{Wife}^{\mathcal{J}_3} = \{\mathit{girl}\}, & \mathit{Priest}^{\mathcal{J}_3} = \{\mathit{john}, \mathit{pedro}, \mathit{ivan}\}, & \mathit{HasHusb}^{\mathcal{J}_3} = \{(\mathit{girl}, \mathit{guy})\},
\end{array}$$

where $guy \in \Delta \setminus \mathsf{adom}(\mathcal{K}_2) \setminus \{girl\}$ is an element of the domain.

Indeed, all $\mathcal{J}_i$'s satisfy $\mathcal{N}_1$ and $\mathcal{T}_2$. To see that they are in $\mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}_2, \mathcal{N}_1)$, observe that every model $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}_2, \mathcal{N}_1)$ can be obtained from $\mathcal{I}$ by making modifications that guarantee that $\mathcal{J} \models \mathcal{N}_1 \cup \mathcal{K}_2$ and that the distance between $\mathcal{I}$ and $\mathcal{J}$ is minimal. What are these modifications? Clearly, $Priest(john)$ should hold in $\mathcal{J}$. Moreover, no priest can be in the $HasHusb$ relation since $(Priest \sqsubseteq \neg\exists HasHusb^-) \in \mathcal{T}_2$. Hence, $john$ cannot be in the $HasHusb$ relation with $girl$ after evolution, and the first necessary modification in $\mathcal{I}$ is to drop the atom $HasHusb(girl, john)$ and to add the atom $Priest(john)$:

$$\mathcal{J}' = (\mathcal{I} \setminus \{HasHusb(girl, john)\}) \cup \{Priest(john)\}.$$

Observe that this modification is not enough, i.e., $\mathcal{J}' \notin \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}_2, \mathcal{N}_1)$, since $\mathcal{J}'$ does not satisfy the TBox, namely, the assertion $Wife \sqsubseteq \exists HasHusb$. Indeed, $girl$ is still a wife in $\mathcal{J}'$, while there is no husband for her, that is, no atom of the form $HasHusb(girl, x)$ for any x is in $\mathcal{J}'$. This problem can be solved by either dropping $Wife(girl)$ from $\mathcal{J}'$ or by finding her a husband, that is, adding $HasHusb(girl, x)$ to $\mathcal{J}'$ for some x. The model $\mathcal{J}_0$ corresponds to the former option, that is:

$$\mathcal{J}_0 = \mathcal{J}' \setminus \{Wife(girl)\}. \tag{16}$$

and the other three $\mathcal{J}_i$'s correspond to the latter one.

Regarding the other option, who should be the husband x of $girl$ in $\mathcal{J}$? There are two possibilities in general: the husband is either one of the two priests (i.e., $x = pedro$ or $x = ivan$), or some other person (i.e., $x = guy$). Clearly, if a priest, say $pedro$, is a husband of $girl$ in $\mathcal{J}$, then he should quit the priesthood due to the TBox assertion $Priest \sqsubseteq \neg HasHusb^-$, i.e., $Priest(pedro)$ should not be in $\mathcal{J}$. Thus, further modifications corresponding to these possibilities give exactly the models $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ defined above:

$$\begin{aligned}
\mathcal{J}_1 &= (\mathcal{J}_0 \setminus \{Priest(pedro)\}) \cup (\{HasHusb(girl, pedro)\} \cup \{Wife(girl)\})\,, && (17)\\
\mathcal{J}_2 &= (\mathcal{J}_0 \setminus \{Priest(ivan)\}) \cup (\{HasHusb(girl, ivan)\} \cup \{Wife(girl)\})\,. && (18)\\
\mathcal{J}_3 &= (\mathcal{J}_0 \setminus \emptyset) \cup (\{HasHusb(girl, guy)\} \cup \{Wife(girl)\})\,.
\end{aligned}$$

Note that we wrote the three formulas above in a specific way: first we subtract atoms about $Priest$ from $\mathcal{J}_0$ (whenever it is needed), and then we add $HasHusb$ and $Wife$-atoms that are required to comply with the TBox $\mathcal{T}_2$. This is done in order to be coherent with the BP procedure which we present later in this section (see Section 3.4.3). ∎

Lack of Canonical Models Recall that for every $DL\text{-}Lite_{core}$ KB $\mathcal{K}$, the set $\mathsf{Mod}(\mathcal{K})$ has a canonical model. At the same time, continuing with Example 3.9, one can verify that any model $\mathcal{J}_{can}$ that can be homomorphically embedded into the four $\mathcal{J}_i$s is such that $Wife^{\mathcal{J}_{can}} = HasHusb^{\mathcal{J}_{can}} = \emptyset$, and $pedro \notin Priest^{\mathcal{J}_{can}}$ and $ivan \notin Priest^{\mathcal{J}_{can}}$. It is easy to check that any such $\mathcal{J}_{can}$ is *not* in $\mathcal{K}_1 \diamond_S \mathcal{N}_1$ with $S = \mathbf{L}^a_{\subseteq}$. Thus, there is no canonical model in $\mathcal{K}_2 \diamond_S \mathcal{N}_1$ with $S = \mathbf{L}^a_{\subseteq}$ and this set is not expressible in $DL\text{-}Lite_{core}$. This gives us the first reason why $DL\text{-}Lite_{core}$ is not closed under $\mathbf{L}^a_{\subseteq}$-evolution.

Local Functionality Another problem with models $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$ is that they satisfy a special kind of functionality constraints on roles.

Definition 3.9 (Local Functionality)**.** Let R be a role and c a constant; then we call *local functionality of R on c* the formula

$$\mathsf{func}(R, c) = \forall x \forall y.\, (R(x, c) \land R(x, y) \to y = c)\,.$$

Example 3.10. Continuing with Example 3.9, one can see that $\mathcal{K}_2 \diamond \mathcal{N}_1$ satisfies local functionality of $HasHusb$ on both priests $pedro$ and $ivan$, for example:

$$\mathsf{func}(HasHusb, pedro) = \forall x \forall y.[(HasHusb(x, pedro) \land HasHusb(x, y) \to (y = pedro))\,.$$

That is, if in $\mathcal{J} \in \mathcal{K}_2 \diamond \mathcal{N}_1$ either *pedro* or *ivan* is a husband of *girl*, then she cannot be married to anyone else. For example, the following model $\mathcal{J}'$, which violates the local functionality, is *not* in $\mathcal{K}_2 \diamond \mathcal{N}_1$ since it is not minimally distant from $\mathcal{I}$ (or any other model of $\mathcal{K}_2$):

$$\mathit{Wife}^{\mathcal{J}'} = \{\mathit{girl}\}, \quad \mathit{Priest}^{\mathcal{J}'} = \{\mathit{john}, \mathit{ivan}\}, \quad \mathit{HasHusb}^{\mathcal{J}'} = \{(\mathit{girl}, \mathit{pedro}), (\mathit{girl}, \mathit{guy})\}.$$

To see this, one can check that it holds that $\mathcal{I} \ominus \mathcal{J}_1 \subset \mathcal{I} \ominus \mathcal{J}'$ for any model $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$.

At the same time, if *girl* has a husband in $\mathcal{J}$ who is neither *pedro* nor *ivan* she can be married to more people. For example, the following model $\mathcal{J}''$ is in $\mathcal{K}_2 \diamond \mathcal{N}_1$:

$$\mathit{Wife}^{\mathcal{J}''} = \{\mathit{girl}\}, \quad \mathit{Priest}^{\mathcal{J}''} = \{\mathit{john}, \mathit{pedro}, \mathit{ivan}\}, \quad \mathit{HasHusb}^{\mathcal{J}''} = \{(\mathit{girl}, \mathit{guy}_1), (\mathit{girl}, \mathit{guy}_2)\}. \blacksquare$$

The following proposition shows that local functionality is not expressible in *DL-Lite$_{core}$*

Proposition 3.17. *Let R be a role and c a constant. Then $\mathcal{K} \not\models \mathsf{func}(R, c)$ for every DL-Lite$_{core}$ KB $\mathcal{K}$ such that $\mathcal{K} \not\models \neg\exists R^-(c)$.*

As a corollary of the proposition above, since the set $\mathcal{K}_2 \diamond_S \mathcal{N}_1$ with $S = \mathbf{L}^a_\subseteq$ satisfies local functionality, it is not expressible in *DL-Lite$_{core}$*. This gives us the second argument why *DL-Lite$_{core}$* is not closed under $\mathbf{L}^a_\subseteq$-evolution.

Dually-Affected Roles Both lack of canonical models and local functionality for $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ observed above are due to *dual-affection* and *triggering* defined as follows:

Definition 3.10 (Dual-Affection and Triggering)**.** Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a *DL-Lite$_{core}$*-evolution setting. Then a role R is *dually-affected* in $\mathcal{T}$ if there are atomic concepts A_1 and A_2 such that $\mathcal{T} \models A_1 \sqsubseteq \exists R$ and $\mathcal{T} \models \exists R^- \sqsubseteq \neg A_2$. A dually-affected role R is *triggered* is $\mathcal{E}$ if $\mathcal{A} \not\models_{\mathcal{T}} \neg\exists R^-(b)$ and $\mathcal{N} \models_{\mathcal{T}} \neg\exists R^-(b)$, for some constant $b \in \mathsf{adom}(\mathcal{E})$. $\blacksquare$

Example 3.11. In Example 3.9 the role *HasHusb* is dually-affected in $\mathcal{T}_2$. Indeed, $\mathcal{T}_2 \models \mathit{Wife} \sqsubseteq \mathit{HasHusb}$ and $\mathcal{T}_2 \models \exists \mathit{HasHusb}^- \sqsubseteq \neg \mathit{Priest}$. This role is also triggered in $\mathcal{K}_2, \mathcal{N}_1$ since $\mathcal{A}_2 \not\models_{\mathcal{T}_2} \exists \mathit{HasHusb}^-(\mathit{john})$, and $\mathcal{N}_1 \models_{\mathcal{T}_2} \neg\exists \mathit{HasHusb}^-(\mathit{john})$. $\blacksquare$

The following theorem shows that if there is a dually-affected role, we can always find $\mathcal{A}$ and $\mathcal{N}$ to trigger it and thus, to guarantee that $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ is inexpressible in *DL-Lite$_{core}$*.

Theorem 3.18. *Let $\mathcal{T}$ be a DL-Lite$_{core}$ TBox and R a role dually-affected in $\mathcal{T}$. Then there are ABoxes $\mathcal{A}$ and $\mathcal{N}$ such that $(\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ is a DL-Lite$_{core}$-evolution setting and $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ is inexpressible in DL-Lite$_{core}$.*

Proof. By definition, there are concepts A and C such that $\mathcal{T} \models A \sqsubseteq \exists R$ and $\mathcal{T} \models \exists R^- \sqsubseteq \neg C$. Now it is enough to take $\mathcal{A}$ and $\mathcal{N}$ analogous to $\mathcal{A}_2$ and $\mathcal{N}_1$ from Example 3.9, respectively. Then $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ is non-axiomatizable in *DL-Lite$_{core}$* since it has no canonical model and entails local functionality which by Proposition 3.17 prevents expressibility in *DL-Lite$_{core}$*. $\square$

3.4.2 Prototypes

As we discussed above, the set of models $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ may not have a canonical model. A closer look at this $\mathcal{K} \diamond_S \mathcal{N}$ gives a surprising result: this set can be divided (but in general not partitioned) into a finite number of subsets $X_0, \ldots, X_n$, that is, $X_i \subseteq \mathcal{K} \diamond_S \mathcal{N}$ with $i \in \{1, \ldots, n\}$ and $\bigcup_{i=1}^n X_i = \mathcal{K} \diamond_S \mathcal{N}$, where each X_i includes its canonical model $\mathcal{J}_i$. Each of this $\mathcal{J}_i$ is a minimal element in $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ w.r.t. homomorphisms.

Definition 3.11 (Prototypal Set)**.** Let $\mathcal{M}$ be a set of models. A *prototypal set for $\mathcal{M}$* is a finite subset $\mathfrak{J} = \{\mathcal{J}_0, \ldots, \mathcal{J}_n\}$ of $\mathcal{M}$ satisfying the following: for every $\mathcal{J} \in \mathcal{M}$ there exists $\mathcal{J}_i \in \mathfrak{J}$ such that $\mathcal{J}_i \hookrightarrow \mathcal{J}$. We call each $\mathcal{J}_i$ in $\mathfrak{J}$ a *prototype for $\mathcal{M}$*.

The notion of prototypes generalizes the notion of canonical model: for example, if $\mathcal{K}$ is a *DL-Lite*$_{core}$ KB, then a prototypal set $\mathfrak{J}$ for $\mathsf{Mod}(\mathcal{K})$ is equal to $\{\mathcal{I}_{\mathsf{can}}\}$. Clearly, an arbitrary set of models may not have a prototypal set.

Definition 3.12 (Prototypal Set for Evolution Settings)**.** $\mathfrak{J}$ is an *S-prototypal set for an evolution setting* $(\mathcal{K}, \mathcal{N})$ with S a model-based semantics if it is a prototypal set for $\mathcal{K} \diamond_S \mathcal{N}$.

Since we will study only $\mathbf{L}^a_{\subseteq}$-prototypal sets, in the following we will refer to them as *prototypal sets for* $(\mathcal{K}, \mathcal{N})$ and omit the $\mathbf{L}^a_{\subseteq}$ prefix.

Example 3.12. Continuing with Example 3.9, one can check that the sets $X = \{\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_4\}$ and $Y = \{\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4\}$ are prototypal for $(\mathcal{K}_2, \mathcal{N}_1)$, where $\mathcal{J}_0, \ldots, \mathcal{J}_3$ are as in Example 3.9, and $\mathcal{J}_4$ is:

$$\mathcal{J}_4\colon\quad \mathit{Wife}^{\mathcal{J}_4} = \{\mathit{girl}_1, \mathit{girl}_2\}, \quad \mathit{Priest}^{\mathcal{J}_4} = \{\mathit{john}\}, \quad \mathit{HasHusb}^{\mathcal{J}_4} = \{(\mathit{girl}_1, \mathit{pedro}), (\mathit{girl}_2, \mathit{ivan})\}.$$

Note that $X = Y \setminus \{\mathcal{J}_3\}$, i.e., $\mathcal{J}_3$ is not needed in the prototypal set X. This holds due to the fact that $\mathcal{J}_0 \subsetneq \mathcal{J}_3$ and $\mathcal{J}_0$ is homomorphically embeddable in $\mathcal{J}_3$. At the same time, if we drop any model from X, then the resulting set of models it not a prototypal for $(\mathcal{K}_2, \mathcal{N}_1)$ anymore.

Observe that the prototypes $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$ were obtained by manipulations with the model $\mathcal{I}$ from Example 3.9, while $\mathcal{J}_4$ can be obtained from the interpretation $\mathcal{I}' \models \mathcal{K}_2$:

$$\mathcal{I}'\colon\quad \mathit{Wife}^{\mathcal{I}'} = \{\mathit{girl}_1, \mathit{girl}_2\}, \quad \mathit{Priest}^{\mathcal{I}'} = \{\mathit{pedro}, \mathit{ivan}\}, \quad \mathit{HasHusb}^{\mathcal{I}'} = \{(\mathit{girl}_1, \mathit{john}), (\mathit{girl}_2, \mathit{john})\}.$$

Finally, observe that $X \subseteq \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}', \mathcal{T}_2, \mathcal{N}_1)$. ∎

We will show later in this section that for *every DL-Lite*$_{core}$ evolution setting $\mathcal{E}$ there is a prototypal set of size exponential in $|\mathcal{E}|$. To this effect we will present a procedure BP[6] that, given $\mathcal{E}$, constructs such a prototypal set. For the ease of exposition of BP, we consider a restricted form of evolution settings.

Definition 3.13 (Simple Evolution Setting)**.** A *DL-Lite*$_{core}$ evolution setting $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ is *simple*, if

(i) for every role R there is *no* atomic concept A such that $\mathcal{T} \models \exists R \sqsubseteq A$,

(ii) for every two different roles R and R', neither $\mathcal{T} \models \exists R \sqsubseteq \exists R'$ nor $\mathcal{T} \models \exists R \sqsubseteq \neg\exists R'$ holds,

(iii) if $\mathcal{N} \models_{\mathcal{T}} \exists R(a)$, then $\mathcal{N} \models_{\mathcal{T}} R(a, b)$ for some constant b, and

(iv) if for an atomic concept D there is B and R such that $B \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq \neg D$ are in $\mathsf{cl}(\mathcal{T})$, then for every R' it holds that $D \sqsubseteq \exists R' \notin \mathsf{cl}(\mathcal{T})$.

These four restrictions allow us to analyze evolution that affects roles independently for every role. We will comment later on how the following techniques can be extended to the case of general *DL-Lite*$_{core}$-evolution settings.

3.4.3 Procedure BP for Build Prototypal Sects for Evolution Settings

We now introduce the procedure $\mathsf{BP}(\mathcal{E})$ that takes a *DL-Lite*$_{core}$-evolution setting $\mathcal{E}$ as input and returns the prototypal set for $(\mathcal{K}, \mathcal{N})$.

Components of BP Procedure Before introducing to BP, we will introduce several notions and notations that the procedure is based upon. We start with the notion of alignment for models.

Definition 3.14. ($\mathcal{T}$-alignment of a model with an ABox) Let $\mathcal{T}$ be a *DL-Lite*$_{core}$ TBox, $\mathcal{N}$ an ABox with only positive assertions and satisfiable with $\mathcal{T}$, and $\mathcal{I}$ a model of $\mathcal{T}$. Then a *$\mathcal{T}$-alignment of $\mathcal{I}$ with $\mathcal{N}$*, denoted $\mathsf{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$, is defined as follows:

$$\mathsf{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N}) = \mathcal{I} \setminus \bigcup_{g \in \mathcal{I} \text{ s.t. } \{g\} \cup \mathcal{N} \models_{\mathcal{T}} \bot} \mathsf{root}_{\mathcal{T}}(g).$$

[6] BP stands for *Build Prototypes*.

In Example 3.9, the only atom g of $\mathcal{I}$ such that $\{g\} \cup \mathcal{N}_1 \models_{\mathcal{T}} \bot$ is $g = HasHusb(girl, john)$; then, $\mathsf{root}_{\mathcal{T}_2}(g) = \{HasHusb(girl, john),\ Wife(girl)\}$; thus, $\mathsf{Align}_{\mathcal{T}_2}(\mathcal{I}, \mathcal{N}_1) = \{Priest(pedro), Priest(ivan)\}$.

The next definition introduces the set of triggered roles.

Definition 3.15 (Set of Triggered Roles)**.** Let $(\mathcal{K}, \mathcal{N})$ be a simple *DL-Lite*$_{core}$-evolution setting; then, $\mathsf{TR}[\mathcal{K}, \mathcal{N}]$ (where TR stands for *triggered roles*), or simply TR when $\mathcal{K}$ and $\mathcal{N}$ are clear, be the set of all roles dually-affected in $(\mathcal{K}, \mathcal{N})$.

In Example 3.9, $\mathsf{TR}[\mathcal{K}_2, \mathcal{N}_1] = \{HasHusb\}$. The next definition introduces the notion of disjoint atoms.

Definition 3.16 (Disjoint Atoms)**.** Let R be dually-affected and triggered in a simple *DL-Lite*$_{core}$-evolution setting $(\mathcal{K}, \mathcal{N})$. Then, the set of unary atoms $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R) \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ (where DjnAts stands for *Disjoint Atoms*) contains $D(c)$ if $\mathcal{T}$ entails that the range of R is disjoint with D, while $\mathcal{N}$ "says" nothing about $D(c)$. Formally:

$$\begin{aligned}\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R) = \{D(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \mid & R \in \mathsf{TR}[\mathcal{K}, \mathcal{N}],\ \{\exists R^-(c), D(c)\} \models_{\mathcal{T}} \bot, \\ & \mathcal{N} \not\models_{\mathcal{T}} D(c),\ \text{and } \mathcal{N} \not\models_{\mathcal{T}} \neg D(c)\}.\end{aligned}$$

In Example 3.9, $\mathsf{DjnAts}[\mathcal{K}_2, \mathcal{N}_1](HasHusb) = \{Priest(pedro), Priest(ivan)\}$. The set of disjoint atoms for the entire KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$, denoted $\mathsf{DjnAts}(\mathcal{K}, \mathcal{N})$, or DjnAts when the parameters are clear, is $\bigcup_{R \in \mathsf{TR}} \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)$. The next definition introduces immediate subconcepts.

Definition 3.17 (Immediate Sub-Concept)**.** For a role R, the set $\mathtt{ISubCon}[\mathcal{T}](\exists R)$ (where `ISubCon` stands for *Immediate Sub-Concepts*) is the set of atomic concepts that are subsumed by $\exists R$ and are "immediately" under $\exists R$ in the concept hierarchy generated by $\mathcal{T}$. Formally:

$$\mathtt{ISubCon}[\mathcal{T}](\exists R) = \{A \mid \mathcal{T} \models A \sqsubseteq \exists R \text{ and there is no } A' \neq A \text{ s.t. } \mathcal{T} \models A \sqsubseteq A' \text{ and } \mathcal{T} \models A' \sqsubseteq \exists R\}.$$

In Example 3.9, $\mathtt{ISubCon}[\mathcal{T}_2](\exists HasHusb) = \{Wife\}$.

We are ready to proceed to the description of the BP procedure. It works similar to the way we described in Equations (16)-(18) of Example 3.9, that is, by first constructing one prototype $\mathcal{J}_0$ by "aligning" $\mathcal{I}_{\mathsf{can}}$ of $\mathcal{K}$ with $\mathcal{N}$ (recall that in $\mathcal{J}_0$ of Example 3.9 *girl* is not a *Wife* anymore and all the priests of $\mathcal{I}$ remain priests), and then manipulating $\mathcal{J}_0$ in order to get all the other prototypes. We will further refer to such a model $\mathcal{J}_0$ as the *zero prototype.* We start with a procedure BZP for constructing the zero prototype.

Procedure BZP for Building Zero Prototype The procedure $\mathsf{BZP}(\mathcal{E})$[7] in Figure 4 constructs the zero prototype $\mathcal{J}_0$ for a simple *DL-Lite*$_{core}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$. It works as follows. First, it deletes from the canonical model $\mathcal{I}_{\mathsf{can}}$ of $\mathcal{K}$ all the atoms that are not $\mathcal{T}$-satisfiable with $\mathcal{N}$ (Step 1). Then, in Step 2 the procedure does the following: from the interpretation $\mathcal{J}_0$ resulting in Step 1 it deletes all atoms of the form $A(a)$ (together with the atoms that $\mathcal{T}$-entail $A(a)$) for which there is no constant b from $\mathsf{adom}(\mathcal{K} \cup \mathcal{N})$ such that $\mathcal{J}_0 \models R(a, b)$ or $\mathcal{N} \models_{\mathcal{T}} R(a, b)$. Moreover, it further deletes from $\mathcal{J}_0$ all atoms of the form $R(a, x)$ where $x \in \Delta \setminus \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$. Intuitively, Step 2 works as follows: if neither $\mathcal{J}_0$ nor $\mathcal{N}$ entails an atom of the form $R(a, b)$ (i.e., there is no active-domain husband b of a *girl* provided by $\mathcal{J}_0$ or $\mathcal{N}_1$ in Example 3.9), then the zero prototype should not contain $A(a)$ (i.e., then *girl* stops to be a wife in Example 3.9) and also all atoms $R(a, x)$ for some non-active x. Step 3 combines $\mathcal{J}_0$ resulting from Step 2 with $\mathcal{N}$ and chases them in order to obtain a model of $\mathcal{T} \cup \mathcal{N}$. Finally, Step 4 returns $\mathcal{J}_0$.

We illustrate BZP on the following example.

[7] BZP stands for *Build Zero Prototype.*

$\mathsf{BZP}(\mathcal{E})$

1. $\mathcal{J}_0 := \mathsf{Align}_\mathcal{T}(\mathcal{I}_{\mathsf{can}}, \mathcal{N})$, where $\mathcal{I}_{\mathsf{can}}$ is the canonical model of $\mathcal{K}$;
2. For each $R \in \mathsf{TR}[\mathcal{K}, \mathcal{N}]$ do
   for each $A \in \mathtt{ISubCon}[\mathcal{K}](\exists R)$ do
   if $A(x) \in \mathcal{J}_0$ for some $x \in \Delta$, and for every $b \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$: $\mathcal{J}_0 \not\models_\mathcal{T} R(x,b)$, $\mathcal{N} \not\models_\mathcal{T} R(x,b)$
   then $\mathcal{J}_0 := \mathcal{J}_0 \setminus (\mathsf{root}_\mathcal{T}(A(x)) \cup \bigcup_{y \in \Delta \setminus \mathsf{adom}(\mathcal{K})} \{R(x,y)\})$;
3. $\mathcal{J}_0 := \mathsf{chase}_\mathcal{T}(\mathcal{J}_0 \cup \mathcal{N})$;
4. Return $\mathcal{J}_0$.

Figure 4 – $\mathsf{BZP}(\mathcal{E})$ procedure for building the zero prototype $\mathcal{J}_0$ for a simple *DL-Lite*$_{core}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$

Example 3.13. In Example 3.9, the zero prototype obtained by $\mathsf{BZP}(\mathcal{K}_2, \mathcal{N}_1)$ is $\mathcal{J}_0$. Indeed, the canonical model of $\mathcal{K}_2$ is $\mathcal{I}_{\mathsf{can}} = \{Priest(pedro), Priest(ivan), HasHusb(x, john)\}$. Step 1 of $\mathsf{BZP}(\mathcal{K}_1, \mathcal{N}_1)$ returns $\mathcal{I}_{\mathsf{can}} \setminus \{HasHusb(x, john)\}$ and Step 2 does nothing. Finally, Step 3 returns the interpretation $\mathsf{chase}(\{Priest(pedro), Priest(ivan)\} \cup \{Priest(john)\})$, which coincides with $\mathcal{J}_0$ of Example 3.9.

Consider another example: $\mathcal{A} = \{C(a)\}$, $\mathcal{T} = \{C \sqsubseteq A, A \sqsubseteq \exists R, \exists R^- \sqsubseteq \neg B\}$, and $\mathcal{N} = \{B(b)\}$. Then, $\mathcal{I}_{\mathsf{can}} = \{C(a), A(a), R(a,x)\}$. Step 1 of $\mathsf{BZP}(\mathcal{K}, \mathcal{N})$ returns the model $\mathcal{I}_{\mathsf{can}}$; Step 2 deletes from $\mathcal{I}_{\mathsf{can}}$ the atom $R(a,x)$ and $\mathsf{root}_\mathcal{T}(A(a)) = \{C(a), A(a)\}$, that is, it returns $\emptyset$; finally, Step 3 returns $\mathcal{J}_0 = \mathsf{chase}_\mathcal{T}\{\emptyset \cup \{B(b)\}\} = \{B(b)\}$. ■

Procedure BP for Building Prototypes The procedure $\mathsf{BP}(\mathcal{E})$ for constructing $\mathfrak{J}$ (see Figure 5) takes a simple *DL-Lite*$_{core}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ as input, constructs the zero prototype $\mathcal{J}_0$ by calling BZP (at Step 1), and based on $\mathcal{J}_0$ builds the other prototypes of $\mathfrak{J}$ (Step 2). Each element in $\mathfrak{J}$ corresponds to a distinct triple consisting of a set $\mathcal{D}$ and two tuples $\mathcal{R}$ (depending on $\mathcal{D}$) and $\mathcal{B}$ (depending on $\mathcal{R}$) that are constructed from $\mathcal{K}$ and $\mathcal{N}$. Thus, BP first chooses a triple $\mathcal{D}$, $\mathcal{R}$, $\mathcal{B}$ that is composed of

(i) a set $\mathcal{D}$ of disjoint atoms from $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ (in Example 3.9, $\mathcal{D}$ is any subset of the priests from $\mathcal{A}$, that is, of $\{Priest(pedro), Priest(ivan)\}$);

(ii) a tuple $\mathcal{R}$ of roles R, one for each $D(c) \in \mathcal{D}$, such that $D(c)$ is a disjoint atom for R, that is, $D(c) \in \mathsf{DjnAts}(R)$ (in Example 3.9, $\mathcal{R} = \langle HasHusb \rangle$ for every possible $\mathcal{D}$ since $D(c)$ can be either $Priest(pedro)$ or $Priest(ivan)$ and it holds that $Priest(pedro) \in \mathsf{DjnAts}(HasHusb)$ and $Priest(ivan) \in \mathsf{DjnAts}(HasHusb)$);

(iii) a tuple $\mathcal{B}$ of immediate subconcepts A of $\exists R$ for each $R \in \mathcal{R}$ (in Example 3.9, $\mathcal{B} = \langle Wife \rangle$ since $Wife$ is the only immediate subconcept of $HasHusb$).

Then, it

(a) deletes from $\mathcal{J}_0$ all the atoms $D(c)$ of $\mathcal{D}$ (that is, it deletes c from $D^{\mathcal{J}_0}$) together with all the atoms that $\mathcal{T}$-entail them. In Equations (17) and (18) of Example 3.9; this corresponds to

$$\mathcal{J}_0 \setminus \{Priest(pedro)\} \quad \text{and} \quad \mathcal{J}_0 \setminus \{Priest(ivan)\};$$

(b) adds to what remains from $\mathcal{J}_0$ the chase of pairs of atoms of the form $R(x,c)$, $A(x)$, that is, it connects with R some elements x of $\Delta \setminus \mathsf{adom}(\mathcal{K})$ to the constants c. In Equations (17) and (18) of Example 3.9; this respectively corresponds to adding

$$\{HasHusb(girl, ivan)\} \cup \{Wife(girl)\} \quad \text{and} \quad \{HasHusb(girl, guy)\} \cup \{Wife(girl)\}.$$

Note that $\mathsf{BZP}(\mathcal{E}) \subseteq \mathsf{BP}(\mathcal{E})$ and $\mathcal{J}_0$ corresponds to $\mathcal{J}[\emptyset, \varepsilon, \varepsilon]$, where ε is the empty tuple.

Before we proceed to the main result of this section, i.e., to the proof that for a simple *DL-Lite*$_{core}$-evolution settings $\mathcal{E}$, the set $\mathsf{BP}(\mathcal{E})$ is prototypal for $\mathcal{E}$, we present a number of technical lemmas, propositions, and observations that will help us to proof this result.

Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a simple *DL-Lite*$_{core}$-evolution setting where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{S} = \mathbf{L}^a_\subseteq$. Let $\mathcal{J}_0$ be the zero prototype for $\mathcal{E}$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ a prototype for $\mathcal{E}$ as defined in BP procedure.

$\mathsf{BP}(\mathcal{E})$

1. $\mathcal{J}_0 := \mathsf{BZP}(\mathcal{E})$;
2. For each set $\mathcal{D} = \{D_1(c_1), \ldots, D_k(c_k)\} \subseteq \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ do
   for each vector $\mathcal{R} = \langle R_1, \ldots, R_k\rangle$, s.t. $R_j \in \mathsf{TR}$ and $D_j(c_j) \in \mathsf{DjnAts}(R_j)$ for $j \in \{1, \ldots, k\}$ do
   for each vector $\mathcal{B} = \langle A_1, \ldots, A_k\rangle$ s.t. $A_j \in \mathtt{ISubCon}[\mathcal{T}](\exists R_j)$ for $j \in \{1, \ldots, k\}$ do
   $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] := \Big(\mathcal{J}_0 \setminus \bigcup_{i=1}^{k} \mathsf{root}_{\mathcal{T}}(D_i(c_i))\Big) \cup \bigcup_{i=1}^{k} \mathsf{chase}_{\mathcal{T}}(\{R_i(x_i, c_i), A_i(x_i)\})$,
   where $\{x_1, \ldots, x_k\}$ are pairwise distinct constants from $\Delta \setminus \mathsf{adom}(\mathcal{K})$ and fresh for $\mathfrak{J}$;
   $\mathfrak{J} := \mathfrak{J} \cup \{\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]\}$.
3. Return $\mathfrak{J}$;

Figure 5 – $\mathsf{BP}(\mathcal{E})$ procedure of building the prototypal set $\mathfrak{J}$ for a simple *DL-Lite*$_{core}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$

We now exhibit models $\mathcal{I}_0 \models \mathcal{K}$ and $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{K}$ which can be considered as "preimages" of $\mathcal{J}_0$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$, respectively, in the sense that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}], \mathcal{T}, \mathcal{N})$.

Lemma 3.19. *Let $S = \mathbf{L}^a_{\subseteq}$ and $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a simple DL-Lite$_{core}$-evolution setting with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$. For the model $\mathcal{J}_0$ returned by $\mathsf{BZP}(\mathcal{E})$ and every other model $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ returned by $\mathsf{BP}(\mathcal{E})$, it holds that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}], \mathcal{T}, \mathcal{N})$, where $\mathcal{I}_0$ and $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ are models of $\mathcal{K}$ defined as following:*

$$\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] = \mathcal{I}_{\mathsf{can}} \cup \bigcup_{1 \leq i \leq |\mathcal{D}|} \mathsf{chase}_{\mathcal{T}}\Big(\{R_i(x_i, d), A_i(x_i) \mid R_i \in \mathcal{R},\ A_i \in \mathcal{B},\ \mathcal{N} \models_{\mathcal{T}} \neg\exists R^-(d), \mathcal{A} \not\models_{\mathcal{T}} \neg\exists R^-(d)\}\Big), \tag{19}$$

$$\mathcal{I}_0 = \mathsf{chase}_{\mathcal{T}}\Big(\mathcal{A} \cup \bigcup_{A(a) \in \mathcal{A}_1} \{R_a(a, b_a) \mid \text{for corresponding } R_a \text{ and } b_a\}\Big), \tag{20}$$

where the auxiliary ABox $\mathcal{A}_1$ is as follows:

$$\mathcal{A}_1 = \{A(a) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \mid \text{there is } R_a \in \mathsf{TR}[\mathcal{K}, \mathcal{N}], \text{ s.t. } A \in \mathtt{ISubCon}[\mathcal{T}](\exists R_a) \text{ and } \forall x \in \Delta\colon \mathcal{N} \not\models_{\mathcal{T}} R_a(a, x),\ \mathcal{A} \not\models_{\mathcal{T}} R_a(a, x) \text{ and there is } b_a \in \Delta\colon \mathcal{N} \models_{\mathcal{T}} \neg\exists R^-(b_a)\},$$

Proof. First, we show that $\mathcal{I}_0 \models \mathcal{T} \cup \mathcal{N}$. Indeed, $\mathcal{I}_0 \models \mathcal{A}$ follows from the definition of chase and Equation (20). To see that $\mathcal{I}_0 \models \mathcal{T}$, observe that, by the definition of $\mathcal{A}_1$, the set $\mathcal{A} \cup \{R_a(a, b_a)\}$ satisfies all the NIs in $\mathsf{cl}(\mathcal{T})$. Moreover, $\{R_a(a, b_a), R_{a'}(a', b_{a'})\}$, where a and a' are such that $A(a)$ and $A'(a')$ are in $\mathcal{A}_1$ for some concepts A and A', satisfies all the NIs in $\mathsf{cl}(\mathcal{T})$. Thus, the set of MAs that is chased in Equation (20) satisfies all the NIs in $\mathsf{cl}(\mathcal{T})$. Hence, due to Lemma 12 of [11], $\mathcal{I}_0 \models \mathcal{T}$.

Now we show that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{T} \cup \mathcal{N}$. The fact that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{A}$ trivially follows from the fact that $\mathcal{I}_{\mathsf{can}} \models \mathcal{A}$. To see that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{T}$, observe that for each i the set $\{R_i(x_i, d), A_i(x_i)\}$ satisfies all the NIs in $\mathsf{cl}(\mathcal{T})$, so does its chase with $\mathcal{T}$. Since x_is are fresh and $(\mathcal{K}, \mathcal{N})$ is a simple *DL-Lite*$_{core}$-evolution setting, for any $g_1 \in \mathsf{chase}_{\mathcal{T}}(\{R_i(x_i, d), A_i(x_i)\})$ and $g_2 \in \mathsf{chase}_{\mathcal{T}}(\{R_j(x_j, d), A_j(x_j)\})$, it holds that $\{g_1, g_2\} \not\models_{\mathcal{T}} \bot$. Therefore, we can apply Proposition 3.2 to the union of chases over $1 \leq i \leq |\mathcal{D}|$ and conclude that it satisfies $\mathcal{T}$. Clearly, for each $g_1 \in \mathcal{I}_{\mathsf{can}}$ and g_2 in the union of chases, $\{g_1, g_2\} \not\models_{\mathcal{T}} \bot$ and again, by applying Proposition 3.2, we conclude that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{T}$.

The proof of $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}], \mathcal{T}, \mathcal{N})$, is straightforward by the definition of BZP and BP procedures. □

3.4.4 Correctness of BP Procedure

Before we proceed to the main result of this section, i.e., to the proof that for a simple *DL-Lite*$_{core}$-evolution settings $\mathcal{E}$, the set $\mathsf{BP}(\mathcal{E})$ is prototypal for $\mathcal{E}$, we present a number of technical lemmas, propositions and observations that will help us to proof this result.

Our next observation is that all $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ share the alignment of $\mathcal{I}$ without disjoint atoms and immediate sub-concepts. In terms of Example 3.9, these $\mathcal{J}'$s share $\mathcal{I}$ without $\mathit{Priest}(\mathit{pedro})$, $\mathit{Priest}(\mathit{ivan})$, and $\mathit{Wife}(\mathit{girl})$.

Lemma 3.20. *Let $S = \mathbf{L}^a_{\subseteq}$ and $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a simple DL-Lite$_{core}$-evolution setting.*
(i) For every $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ it holds that:

$$\mathsf{Align}_{\mathcal{T}}(I \setminus \mathcal{B}_{\mathcal{I}}, \mathcal{N}) \subseteq \mathcal{J}, \quad \text{where} \quad \mathcal{B}_{\mathcal{I}} = \bigcup_{D(c) \in \mathcal{S}} \mathsf{root}^{\mathcal{I}}_{\mathcal{T}}(D(c)) \quad \text{and}$$

$$\begin{aligned}\mathcal{S} = \{D(c) \in \mathcal{I} \mid\ & \mathcal{N} \not\models_{\mathcal{T}} D(c),\ \mathcal{N} \not\models_{\mathcal{T}} \neg D(c), \text{ and there is } R \text{ dually-affected in } \mathcal{K} \text{ s.t.} \\ & \{\exists R^-(c), D(c)\} \models_{\mathcal{T}} \bot, \text{ and there are } x, d \in \Delta \text{ s.t.} \\ & \mathcal{I} \models R(x, d),\ \mathcal{N} \models \neg \exists R^-(d)\}.\end{aligned}$$

(ii) In particular, if $\mathcal{I}$ is $\mathcal{I}_{\mathsf{can}}$, i.e., a canonical model of $\mathcal{K}$, then for every $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_{\mathsf{can}}, \mathcal{T}, \mathcal{N})$ it holds that

$$\mathsf{Align}_{\mathcal{T}}(\mathcal{I}_{\mathsf{can}} \setminus \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}], \mathcal{N}) \subseteq \mathcal{J}.$$

Note that $\mathcal{B}_{\mathcal{I}}$ in the lemma above can be seen as an extension of the set of disjoint atoms $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ from KBs to models of this KBs in the sense that $\mathcal{B}_{\mathcal{I}} \cap \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) = \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$. As a consequence of Lemma 3.20, consider the following definition.

Definition 3.18 (Constant and Variable Parts of Models)**.** For a given $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$, every model $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ can be partitioned in two parts:
(i) a *constant* part $\mathcal{J}_c = \mathsf{Align}_{\mathcal{T}}(\mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}, \mathcal{N})$, which is the same across all elements of $\mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$, and
(ii) a *variable* part $\mathcal{J}_v = \mathcal{J} \setminus \mathcal{J}_c$, which varies from one element of $\mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ to another.
Clearly, $\mathcal{J} = \mathcal{J}_c \cup \mathcal{J}_v$ and $\mathcal{J}_c \cap \mathcal{J}_v = \emptyset$. Note that $\mathcal{J}_c$ is the constant part of the entire set $\mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ in the sense that $\mathcal{J}_c \subseteq \bigcap_{\mathcal{J}' \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})} \mathcal{J}'$.

Before proceeding to the final observation, consider the following technical property which we will use in the proof of this final observation.

Proposition 3.21. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a simple DL-Lite$_{core}$-evolution setting, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$. Let $\mathcal{I} \models \mathcal{K}$ and $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Then,*
(i) if $\mathcal{N} \not\models_{\mathcal{T}} A(a)$ and there is an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ such that $\mathcal{I} \cup \{A(a)\} \not\models \alpha$, then $A(a) \notin \mathcal{J}$.
(ii) if $\mathcal{N} \not\models_{\mathcal{T}} \exists R(a)$, $\mathcal{N} \not\models_{\mathcal{T}} \exists R^-(b)$ and there is an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ s.t. $\mathcal{I} \cup \{R(a, b)\} \not\models \alpha$, then $R(a, b) \notin \mathcal{J}$.

Finally, consider the following property of models $\mathcal{I}$ and $\mathcal{J}$ w.r.t. disjoint atoms where $\mathcal{I}$ and $\mathcal{J}$ are related through $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$.

Lemma 3.22. *Let $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a simple DL-Lite$_{core}$-evolution setting, $\mathcal{I} \models \mathcal{K}$, and $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. If $D(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$, then the following holds:*
(i) If $D(c) \notin \mathcal{J}$, then there exists $R \in \mathsf{TR}$ and $A \in \mathtt{ISubCon}[\mathcal{T}](\exists R)$ s.t. $D(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)$ and

$$\{R(x, c), A(x)\} \subseteq \mathcal{J} \text{ for some } x \in \Delta. \tag{21}$$

(ii) If $D(c) \in \mathcal{J}$, then for every unary MA, an atom $A(c)$ satisfying $\mathcal{K} \models A(c)$, where $\mathcal{T} \models A \sqsubseteq D$ and $A(c) \in \mathsf{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$, the inclusion $A(c) \in \mathcal{J}$ holds.[8]

[8] Recall that the evolution setting $(\mathcal{K}, \mathcal{N})$ is simple and therefore there is no role R such that $\exists R \sqsubseteq D$.

Proof. Case (i): Let $\mathcal{I} \models \mathcal{K}$ be such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Assume that $D(c) \notin \mathcal{J}$, and

$$\{R(x,c), A(x)\} \not\subseteq \mathcal{J} \text{ for each } x \in \Delta, \text{ each } R \in \mathsf{TR} \text{ and each atomic concept } A \text{ such that } D(c) \in \mathsf{DjnAts}[\mathcal{K},\mathcal{N}](R) \text{ and } A \in \mathtt{ISubCon}[\mathcal{T}](\exists R). \quad (22)$$

Observe that the condition $\{R(x,c), A(x)\} \not\subseteq \mathcal{J}$ is satisfied when $R(x,c) \in \mathcal{J}$ and $A(x) \notin \mathcal{J}$. Let

$$\mathcal{D} = \{R(x,c) \mid x \in \Delta,\ R(x,c) \in \mathcal{J},\ D(c) \in \mathsf{DjnAts}[\mathcal{K},\mathcal{N}](R)\}.$$

Assume $\mathcal{D} \neq \emptyset$. Consider $\mathsf{root}^{\mathcal{J}}_{\mathcal{T}}(R(x,c))$. Due to the assumption in Equation (22), there are no unary atoms in $\mathsf{root}^{\mathcal{J}}_{\mathcal{T}}(R(x,c))$. Moreover, since $(\mathcal{K},\mathcal{N})$ is a simple evolution setting there are no binary atoms in $\mathsf{root}^{\mathcal{J}}_{\mathcal{T}}(R(x,c))$ besides $R(x,c)$. Thus, $\mathsf{root}^{\mathcal{J}}_{\mathcal{T}}(R(x,c)) = \{R(x,c)\}$. Consider a model

$$\mathcal{J}' = \mathcal{J} \setminus \bigcup_{R(x,c)\in\mathcal{D}} \{R(x,c)\}.$$

We now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, which contradicts the fact that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Observe that $\mathcal{J}' \models \mathcal{N}$. Assume this is not the case and there is an MA g such that $\mathcal{N} \models g$ and $\mathcal{J}' \not\models g$. We have two cases here.

- Assume that g is a positive MA. Since $\mathcal{J} \models \mathcal{N}$, we conclude that $g \in \bigcup_{R(x,c)\in\mathcal{D}}\{R(x,c)\}$. Therefore, $g = R(x,c)$ for some $R(x,c) \in \mathcal{D}$, we have that $R(x,c) \in \mathcal{N}$. Combining this with the fact that $\{D(c), \exists R^-(c)\} \models_{\mathcal{T}} \perp$ we conclude that $\{D(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \perp$. This contradicts the fact that $\mathcal{N} \not\models_{\mathcal{T}} D(c)$. Thus, $\mathcal{J}' \models \mathcal{N}$.

- Assume that g is a negative MA. Since $\mathcal{J} \models \mathcal{N}$ and $\mathcal{J}' \subseteq \mathcal{J}$, it trivially holds that $\mathcal{J}' \models \mathcal{N}$.

Due to Proposition 3.3, $\mathcal{J}' \models \mathcal{T}$. Therefore, $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$. By construction of $\mathcal{J}'$, we have that every $R(x,c)$ from $\mathcal{D}$ is *not* in $\mathcal{I}$ and $\mathcal{J}'$, while it is in $\mathcal{J}$. Therefore, $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$ and we conclude the proof.

Assume $\mathcal{D} = \emptyset$, then consider

$$\mathcal{J}' = \mathcal{J} \cup \mathcal{I}[D(c)].^9$$

Again, we now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, which contradicts the fact that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. The inclusion $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$ follows from $\mathcal{I}[D(c)] \subseteq \mathcal{I}$, and $\mathcal{I}[D(c)] \subseteq \mathcal{J}'$, and $D(c) \notin \mathcal{J}$. Then, $\mathcal{J}' \models \mathcal{N}$ follows from $\mathcal{J} \models \mathcal{N}$. It remains to show that $\mathcal{J}' \models \mathcal{T}$ holds, and we proceed to a proof of this entailment.

Clearly, both $\mathcal{J}$ and $\mathcal{I}[D(c)]$ satisfy $\mathcal{T}$. Due to Proposition 3.2, to finish the proof of $\mathcal{J}' \models \mathcal{T}$, it remains to show that for every $g_1 \in \mathcal{J}$ and $g_2 \in \mathcal{I}[D(c)]$ it holds that $\{g_1, g_2\} \not\models_{\mathcal{T}} \perp$. Assume this is not the case and there are $g_1 \in \mathcal{J}$ and $g_2 \in \mathcal{I}[D(c)]$ such that $\{g_1, g_2\} \models_{\mathcal{T}} \perp$. Then, there is an NI α in $\mathsf{cl}(\mathcal{T})$ such that $g_1 \to \neg g_2$ is an instantiation of the first-order interpretation of α. Observe that $g_2 \neq D(c)$. Indeed, if $g_2 = D(c)$, then, since $D(c)$ is a unary atom, α is of the form $D \sqsubseteq \neg B$.

(i) If $B = A'$ for some atomic concept A', then $g_1 = A'(c) \in \mathcal{J}$. If $\mathcal{N} \models_{\mathcal{T}} A'(c)$, then $\mathcal{N} \models_{\mathcal{T}} \neg D(c)$, which contradicts the fact that $\mathcal{N} \not\models_{\mathcal{T}} \neg D(c)$. If $\mathcal{N} \not\models_{\mathcal{T}} A'(c)$, then, due to Case (i) of Proposition 3.21 and the facts that $D(c) \in \mathcal{I}$ and $\{D(c), A'(c)\} \models_{\mathcal{T}} \perp$, we conclude that $A'(c) \notin \mathcal{J}$, which gives a contradiction.

(ii) If $B = \exists R_1$ for some role R_1, then $g_1 = R_1(c,y) \in \mathcal{J}$ for some $y \in \Delta$. Assume that $\mathcal{N} \models_{\mathcal{T}} R_1(c,y)$, then $\mathcal{N} \models_{\mathcal{T}} \neg D(c)$, which gives a contradiction. If $\mathcal{N} \not\models_{\mathcal{T}} R_1(c,y)$, and also $\mathcal{N} \not\models_{\mathcal{T}} \exists R_1(c)$, $\mathcal{N} \not\models_{\mathcal{T}} \exists R_1(y)$, then, due to Case (ii) of Proposition 3.21 and the facts that $D(c) \in \mathcal{I}$ and $\{D(c), R'(c,y)\} \models_{\mathcal{T}} \perp$, we conclude that $R'(c,y) \notin \mathcal{J}$ and again obtain a contradiction. If $\mathcal{N} \not\models_{\mathcal{T}} R_1(c,y)$ and either $\mathcal{N} \models_{\mathcal{T}} \exists R_1(c)$ or $\mathcal{N} \models_{\mathcal{T}} \exists R_1^-(y)$ holds, then, due to $D(c) \in \mathcal{I}$ and $\{D(c), R'(c,y)\} \models_{\mathcal{T}} \perp$, one can concludethat $R'(c,y) \notin \mathcal{J}'$, thus we obtain a contradiction.

[9] Recall that $\mathcal{I}[D(c)]$ is a minimal submodel of $\mathcal{I}$ containing $D(c)$.

Since $(\mathcal{K}, \mathcal{N})$ is a simple evolution setting, every $g' \in \mathcal{I}[D(c)]$ is unary [10] and $\mathcal{D}(c) \models_{\mathcal{T}} g'$. Thus, we can apply to such g' the same argument as to $D(c)$ above to obtain a contradiction Thus, due to Proposition 3.2, we conclude that $\mathcal{J} \cup \mathcal{I}[D(c)] \models \mathcal{T}$. This implies that $\mathcal{J} \notin \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$, yields a contradiction, and concludes the proof of Case (i).

Case (ii): Assume there is $D(c) \in \mathcal{J}$ and a unary MA $A(c)$ satisfying $\mathcal{K} \models A(c)$, $\mathcal{T} \models A \sqsubseteq D$, and $A(c) \in \mathsf{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$, while $A(c) \notin \mathcal{J}$ holds. Let $\mathcal{I}$ be a model of $\mathcal{K}$ such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Consider

$$\mathcal{J}' = \mathcal{J} \cup \mathcal{I}[A(c)].$$

Again, we now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, which contradicts the fact that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Observe that $\mathcal{J}' \models \mathcal{N}$, since $\mathcal{J}$ does so. Since both $\mathcal{J}$ and $\mathcal{I}[A(c)]$ satisfy $\mathcal{T}$, then, due to Proposition 3.2 and Lemma 12 of [11], it suffices to show that for every NI $\alpha \in \mathsf{cl}(\mathcal{T})$ and every $g \in \mathcal{J}$ and $f \in \mathcal{I}[A(c)]$, $\{f, g\}$ satisfies α. Assume there is an NI α in $\mathsf{cl}(\mathcal{T})$, $g \in \mathcal{J}$ and $f \in \mathcal{I}[A(c)]$ such that $\{g, f\} \models_{\{\alpha\}} \bot$. If $\mathcal{N} \models_{\mathcal{T}} g$, then $\mathcal{N} \cup \{f\} \models_{\{\alpha\}} \bot$. Thus, $A(c) \in \mathsf{root}^{\mathcal{I}}_{\mathcal{T}}(f)$ and therefore $A(c) \notin \mathsf{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$ which gives a contradiction. Assume $\mathcal{N} \not\models g$. If g is unary, then we can apply Case (ii) of Proposition 3.21 to obtain a contradiction. If g is binary, then, as in the proof of Case (i) of the current proposition, we can apply Case (ii) of Proposition 3.21 or Proposition A.4, and obtain a contradiction. □

The following result establishes the fundamental property of the set of models computed by BP: this set is prototypal for the input evolution setting and exponential in the size of the setting.

Theorem 3.23. *Let $\mathcal{E}$ be a simple DL-Lite$_{core}$-evolution setting. Then, the set $\mathsf{BP}(\mathcal{E})$ is a prototypal set for $\mathcal{E}$. Moreover, $|\mathsf{BP}(\mathcal{E})|$ is exponential in $|\mathcal{E}|$.*

Proof. Let $S = \mathbf{L}^a_{\subseteq}$. To see the bound on $|\mathsf{BP}(\mathcal{E})|$ observe that the number of prototypes is polynomial in the number of triples $\mathcal{D}$, $\mathcal{R}[\mathcal{D}]$ and $\mathcal{B}[\mathcal{R}]$, where the number of different components in each triple is exponential in $|\mathcal{E}|$.

By the definition of prototypal sets, in order to prove that $\mathsf{BP}(\mathcal{E}) = \mathfrak{J}$ is a prototypal set, one should show the following two conditions:

(i) $\mathsf{BP}(\mathcal{E}) \subseteq \mathcal{K} \diamond_S \mathcal{N}$ and

(ii) for every model $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ there exists a model $\mathcal{J}' \in \mathsf{BP}(\mathcal{E})$ such that $\mathcal{J}'$ is homomorphically embeddable into $\mathcal{J}$.

Condition (i) follows from Lemma 3.19. Indeed, for every element $\mathcal{J}$ of $\mathsf{BP}(\mathcal{E})$ the lemma presents a model $\mathcal{I}$ of $\mathcal{K}$ that "evolves" in $\mathcal{J}$, i.e., for $\mathcal{J}_0$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ it presents $\mathcal{I}_0$ and $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ such that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \subseteq \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Therefore $\mathsf{BP}(\mathcal{E}) \subseteq \mathcal{K} \diamond_S \mathcal{N}$.

Now we prove Condition (ii). Let $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$, we will exhibit $\mathcal{J}' \in \mathsf{BP}(\mathcal{E})$ such that $\mathcal{J}' \hookrightarrow \mathcal{J}$. To this effect, consider $\mathcal{D}_{\mathcal{J}}$, the set of all (redundant) atoms from $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ that are *not* in $\mathcal{J}$, that is, $\mathcal{D}_{\mathcal{J}} \subseteq \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ and for every $D(c) \in \mathcal{D}_{\mathcal{J}}$ we have $D(c) \notin \mathcal{J}$, while for every $D(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}] \setminus \mathcal{D}_{\mathcal{J}}$ we have $D(c) \in \mathcal{J}$. Let $\mathcal{R}_{\mathcal{J}} = \langle R^{D(c)} \mid D(c) \in \mathcal{D}_{\mathcal{J}} \rangle$ and $\mathcal{B}_{\mathcal{J}} = \langle A^{D(c)} \mid D(c) \in \mathcal{D}_{\mathcal{J}} \rangle$. Now take

$$\mathcal{J}' := \mathcal{J}[\mathcal{D}_{\mathcal{J}}, \mathcal{R}_{\mathcal{J}}, \mathcal{B}_{\mathcal{J}}],$$

where $\mathcal{J}[\mathcal{D}_{\mathcal{J}}, \mathcal{R}_{\mathcal{J}}, \mathcal{B}_{\mathcal{J}}]$ is constructed by BP as in Figure 5.

Note that *(i)* $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ implies that there exists $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$, and *(ii)* $\mathcal{J}' \in \mathsf{BP}(\mathcal{E})$, due to Condition (i), implies that $\mathcal{J}' \in \mathcal{K} \diamond_S \mathcal{N}$ and therefore there exists $\mathcal{I}' \in \mathsf{Mod}(\mathcal{K})$, which may differ from $\mathcal{I}$, such that $\mathcal{J}' \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}', \mathcal{T}, \mathcal{N})$. Recall that due to Lemma 3.20 we can partition $\mathcal{J}$ (resp., $\mathcal{J}'$) in two parts: a constant $\mathcal{J}_c$ (resp., $\mathcal{J}'_c$) and a variable part $\mathcal{J}_v = \mathcal{J} \setminus \mathcal{J}_c$ (resp., $\mathcal{J}'_v$), i.e., $\mathcal{J} = \mathcal{J}_c \cup \mathcal{J}_v$ (resp., $\mathcal{J}' = \mathcal{J}'_v \cup \mathcal{J}'_v$).

Now, to conclude that $\mathcal{J}' \hookrightarrow \mathcal{J}$ holds, i.e., $(\mathcal{J}'_c \cup \mathcal{J}'_v) \hookrightarrow (\mathcal{J}_c \cup \mathcal{J}_v)$ holds, we will show that there are homomorphisms h_c and h_v such that:

$$\text{(a) } h_c : \mathcal{J}'_c \to \mathcal{J}_c \quad \text{and} \quad \text{(b) } h_v : \mathcal{J}'_v \to \mathcal{J}_v,$$

[10] In particular, Restriction (ii) of Definition 3.13 implies that there is no role Q such that $\mathcal{T} \models \exists Q \sqsubseteq D$.

and the combination of h_c and h_v will be a homomorphism from $\mathcal{J}'$ to $\mathcal{J}$.

- We prove Condition (a) using Lemmas 3.19 and 3.20. Indeed, combining Equation (19) of Lemma 3.19 if $\mathcal{D}_\mathcal{J} \neq \emptyset$ and Equation (20) of Lemma 3.19 otherwise, with Lemma 3.20, Case (ii), we obtain that the constant part $\mathcal{J}'_c$ of $\mathcal{J}'$ is:

$$\mathcal{J}'_c = \mathsf{Align}_\mathcal{T}(\mathcal{I}_{\mathsf{can}} \setminus \mathsf{DjnAts}[\mathcal{K},\mathcal{N}],\mathcal{N})$$

Due to Lemma 3.20, Case (i), we have that the constant part of $\mathcal{J}$ is the following: $\mathcal{J}_c = \mathsf{Align}_\mathcal{T}(\mathcal{I}\setminus\mathsf{DjnAts}[\mathcal{K},\mathcal{N}],\mathcal{N})$. Now recall that by the definition of canonical models, $\mathcal{I}_{\mathsf{can}} \hookrightarrow \mathcal{I}$ holds and this implies that $\mathsf{Align}_\mathcal{T}(\mathcal{I}_{\mathsf{can}} \setminus \mathsf{DjnAts}[\mathcal{K},\mathcal{N}],\mathcal{N}) \hookrightarrow \mathsf{Align}_\mathcal{T}(\mathcal{I} \setminus \mathsf{DjnAts}[\mathcal{K},\mathcal{N}],\mathcal{N})$ holds as well. I.e., there is a homomorphism h_c from $\mathcal{J}'_c$ to $\mathcal{J}_c$.

- We prove Condition (b) using Lemma 3.22. Observe that by the definition of the BP procedure the variable part $\mathcal{J}'_v = \mathcal{J}[\mathcal{D}_\mathcal{J},\mathcal{R}_\mathcal{J},\mathcal{B}_\mathcal{J}]_v$ of $\mathcal{J}'$ is the following:

$$\mathcal{J}'_v = \bigcup_{D(c)\in\mathcal{D}_\mathcal{J}} \{R^{D(c)}(x',c), A^{D(c)}(x') \mid x' \text{ is fresh}\} \cup \tag{23}$$

$$\bigcup_{D(c)\in\mathsf{DjnAts}[\mathcal{K},\mathcal{N}]\setminus\mathcal{D}_\mathcal{J}} \Big(\{D(c)\}\cup\{A(c) \mid \mathcal{K}\models A(c), A(c)\notin\mathcal{D}_\mathcal{J}, A(c)\in\mathsf{AtAlg}(\mathcal{E}), \mathcal{T}\models A\sqsubseteq D\}\Big), \tag{24}$$

where "x' is fresh" means that it is in $\Delta \setminus \mathsf{adom}(\mathcal{K})$, distinct for each set $\{R^{D(c)}(x',c), A^{D(c)}(x')\}$ defined by $D(c)$, and does not appear in $\mathcal{J}'_c$. Also observe that if $\mathcal{D} = \emptyset$, then $\mathcal{J}[\mathcal{D}_\mathcal{J},\mathcal{R}_\mathcal{J},\mathcal{B}_\mathcal{J}]$ is the zero prototype $\mathcal{J}_0$.
It remains to show that there is a homomorphism from $\mathcal{J}'_v$ to $\mathcal{J}_v$. Consider a mapping h_v such that it is the identity mapping on every element of $\mathcal{J}'_v$, but x' (see Equation (23)), and $h_v(x') = x$, where x is from Equation (21) of Lemma 3.22, for every such x'. To see that h_v is a homomorphism, observe how it works on every atom that occurs in Equations (23) and (24):

  (i) on atoms from Equation (23):

$$R^{D(c)}(h_v(x'),h_v(c)) = R^{D(c)}(x,c) \quad \text{and} \quad A^{D(c)}(h_v(x')) = A^{D(c)}(x),$$

  where $R^{D(c)}(x,c) \in \mathcal{J}_v$ and $A^{D(c)}(x) \in \mathcal{J}_v$ hold due to Lemma 3.22, Case (i);

  (ii) for atoms from Equation (24):

$$D(h_v(c)) = D(c) \quad \text{and} \quad A(h_v(c)) = A(c),$$

  where $A(c) \in \mathcal{J}_v$ holds since $A(c)$ satisfies the conditions of Lemma 3.22, Case (ii), and $D(c) \in \mathcal{J}_v$ holds since $D(c) \in \mathsf{DjnAts} \setminus \mathcal{D}_\mathcal{J}$ and consequently $D(c) \in \mathcal{J}$.

Now we consider the following mapping from $\mathcal{J}'$ to $\mathcal{J}$, which we will prove to be a homomorphism:

$$h(x) = \begin{cases} h_c(x) & \text{if } x \text{ appear in } \mathcal{J}'_c, \\ h_v(x) & \text{otherwise.} \end{cases}$$

The correctness of this definition of h follows from the following observation. Note that, by the construction of $\mathcal{J}'$ by the BP procedure, the elements appearing in $\mathcal{J}'$ are either constants (i.e. from $\mathsf{adom}(\mathcal{E})$) or "fresh" elements. The fact that $h_c(a) = h_v(a)$ for a being a constant guarantees the correctness of homomorphic embedding of atoms containing constants, no matter whether an atom with this constant is in $\mathcal{J}'_c$ or $\mathcal{J}'_v$; the fact that "fresh" elements appear only in $\mathcal{J}'_v$ again guarantees that this part will be homomorphic embedded into $\mathcal{J}$ by h correctly, since h_v does so. The elements of Δ which are neither constants nor "fresh" will be homomorphically embedded into $\mathcal{J}$ by h as well, since h_c does so. □

Extension of BP to *DL-Lite*$_{core}$ KBs without Restrictions Results of Theorem 3.23 can be extended to the general case when the evolution setting is not simple. Observe that in the general case the BP procedure does return prototypes but not all of them. Weakening restrictions in Cases (i) and (iii) in the definition of simple evolution settings (i.e., allowing entailments from $\mathcal{K}$ of the form $\exists R \sqsubseteq A$ and $\mathcal{T}$-entailments from $\mathcal{N}$ of the form $\exists R(a)$) results in more than one zero prototype. Weakening in Case (ii) (i.e., allowing entailment from $\mathcal{K}$ of direct role interactions of the form $\exists R \sqsubseteq \exists R'$ and $\exists R \sqsubseteq \neg\exists R'$), leads to the need to iterate BP over constructed prototypes. More precisely, to gain the missing prototypes in this case one should run BP several times (a finite number) iterating over (already constructed) prototypes until no new prototypes can be constructed. Intuitively, the reason is that BP deletes disjoint atoms (atoms of DjnAts) and adds new atoms of the form $R(a, b)$ for some triggered dually-affected role R, which may in turn trigger another dually-affected role, say P, and such triggering may require further modifications, already for P. These further modifications require a new run of BP. For example, if we have $\exists R^- \sqsubseteq \neg\exists P^-$ in the TBox and we set $R(a, b)$ in a prototype, say $\mathcal{J}_k$, this modification triggers role P and we should run BP recursively with the prototype $\mathcal{J}_k$ as if it was the zero prototype. We shall not discuss the general procedures in more details due to space limitations.

Summary of Section 3.4 We discussed why $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ cannot in general be axiomatized in *DL-Lite*$_{core}$, introduced prototypal sets, and a procedure that constructs these sets with exponentially many prototypes for *DL-Lite*$_{core}$-evolution settings.

3.5 Approximating $\mathbf{L}^a_\subseteq$-Evolution

Capturing $\mathbf{L}^a_\subseteq$-Evolution in Richer Logics We start with a discussion on *how* to capture $\mathbf{L}^a_\subseteq$-evolution of *DL-Lite*$_{core}$ KBs in logics richer than *DL-Lite*$_{core}$. As we saw in the previous section, for every *DL-Lite*$_{core}$-evolution setting $(\mathcal{K}, \mathcal{N})$, the evolution result $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ is a finite union of sets of models, $\mathcal{K} \diamond_S \mathcal{N} = \bigcup_i \mathcal{M}_i$, where each $\mathcal{M}_i$ contains a prototype $\mathcal{J}_i$. Thus, for $S = \mathbf{L}^a_\subseteq$ axiomatization of $\mathcal{K} \diamond_S \mathcal{N}$ boils down to axiomatization of each $\mathcal{M}_i$ with some theory Th_i, that is, $\mathcal{M}_i = \mathsf{Mod}(\mathsf{Th}_i)$, and taking the disjunction across these theories. As shown in [34], each Th_i can be computed based on a prototype of $\mathcal{M}_i$ using a *DL-Lite*$_{core}$ KB $\mathcal{K}_i[\mathcal{J}_i]$ (whose canonical model is precisely $\mathcal{J}_i$) and a compensation formula Ψ, which is not expressible in *DL-Lite*$_{core}$, as $\mathsf{Th}_i \equiv \mathcal{K}_i[\mathcal{J}_i] \wedge \Psi$. The compensation formula Ψ cuts off the models which are not in $\mathcal{K} \diamond_S \mathcal{N}$, but cannot be filtered by a *DL-Lite*$_{core}$ KB, e.g., models not satisfying local functionality. It turned out [34] that Ψ is the same for each Th_i, hence:

$$\mathcal{K} \diamond_S \mathcal{N} = \mathsf{Mod}(\Psi \wedge \bigvee_{i=1}^{n} \mathcal{K}[\mathcal{J}_i]),$$

We showed that $\mathsf{Th}_{\mathcal{K} \diamond_S \mathcal{N}} = \Psi \wedge \bigvee_{i=1}^{n} \mathcal{K}[\mathcal{J}_i]$ is in FO[2] and even in $\mathcal{SHOIQ}$ [4].

On the one hand, we do not know how to do $\mathbf{L}^a_\subseteq$-evolution of $\mathcal{SHOIQ}$ KBs: if for $S = \mathbf{L}^a_\subseteq$ one wants to do evolution of the evolution setting $(\mathsf{Th}_{\mathcal{K} \diamond_S \mathcal{N}}, \mathcal{N}_1)$, where $\mathcal{N}_1$ is some new knowledge, then it is still unclear which logic is needed to capture the evolution result and how to do so. On the other hand, we would like to stay within *DL-Lite*$_{core}$, i.e., to return a *DL-Lite*$_{core}$ KB as the evolution result. Therefore, we study now how for $S = \mathbf{L}^a_\subseteq$ to *approximate* $\mathsf{Th}_{\mathcal{K} \diamond_S \mathcal{N}}$ in *DL-Lite*$_{core}$.

Approximating Evolution Results Since for $S = \mathbf{L}^a_\subseteq$ neither the disjunction of $\mathcal{K}_i[\mathcal{J}_i]$ nor Ψ is expressible in *DL-Lite*$_{core}$, one way to approximate $\mathsf{Th}_{\mathcal{K} \diamond_S \mathcal{N}}$ is to take one of $\mathcal{K}_i[\mathcal{J}_i]$. Unfortunately, such an approximation is not sound, that is, for each i there are models of $\mathcal{K}_i[\mathcal{J}_i]$ that are not in $\mathcal{K} \diamond_S \mathcal{N}$. What we propose next is a *DL-Lite*$_{core}$-approximation that is sound and keeps the certain knowledge of $\mathcal{K} \diamond_S \mathcal{N}$, that is, ABox assertions shared by all $\mathcal{K}_i[\mathcal{J}_i]$.

We now formalize the notion of the certain knowledge, w hich is the key component in our *DL-Lite*$_{core}$-approximation of $\mathbf{L}^a_\subseteq$-evolution.

Definition 3.19 (S-Certain MA)**.** Let S be an MBA, $(\mathcal{K}, \mathcal{N})$ an evolution setting, and $\mathcal{K}'$ an S-evolution for $(\mathcal{K}, \mathcal{N})$. Then, a membership assertion g (positive or negative) is S-certain for $(\mathcal{K}, \mathcal{N})$ if $\mathcal{K}' \models g$.

Algorithm 4: $\mathsf{Weeding}(\mathcal{T}, \mathcal{A}, \mathcal{D})$

INPUT : TBox $\mathcal{T}$, and ABoxes $\mathcal{A}$, $\mathcal{D}$, each satisfiable with $\mathcal{T}$
OUTPUT : finite set of membership assertions $\mathcal{A}^w$

1 $\mathcal{A}^w := \mathcal{A}$;
2 **for each** $B_1(c) \in \mathcal{D}$ **do**
3     $\mathcal{A}^w := \mathcal{A}^w \setminus \{B_1(c)\}$ **and**;
4   **for each** $B_2 \sqsubseteq B_1 \in \mathsf{cl}(\mathcal{T})$ **do**
5         $\mathcal{A}^w := \mathcal{A}^w \setminus \{B_2(c)\}$;
6   **if** $B_2(c) = \exists R(c)$ **then**
7   **for each** $R(c,d) \in \mathcal{A}^w$ **do** $\mathcal{D} := \mathcal{D} \cup \{R(c,d)\}$
8   **end**
9   **end**
10 **end**
11 **return** $\mathcal{A}^w$;

Algorithm 5: $\mathsf{ApproxAlg}(\mathcal{E})$

INPUT : *DL-Lite*$_{core}$-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$
OUTPUT : an ABox $\mathcal{A}^{\mathsf{app}}$

1 $\mathcal{A}^{\mathsf{app}} := \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$; $X := \emptyset$;
2 **for each** $R \in \mathsf{TR}[\mathcal{T}, \mathcal{N}]$ **and for each** $a \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$ **do**
3   **if there is no** $R(a,b) \in \mathcal{A}^{\mathsf{app}} \cup \mathcal{N}$ **for some** $b \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$ **then**
4         $X := X \cup \{\exists R(a)\}$;
5   **end**
6 **end**
7 **for each** $A(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ **do** $X := X \cup \{A(c)\}$;
8 **for each** $R \in \mathsf{TR}[\mathcal{T}, \mathcal{N}]$ **do**
9   **for each** $A(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)$ **do** $\mathcal{A}^{\mathsf{app}} := \mathcal{A}^{\mathsf{app}} \setminus \{\neg\exists R^-(c)\}$;
10 **end**
11 $\mathcal{A}^{\mathsf{app}} := \mathcal{N} \cup \mathsf{Weeding}(\mathcal{T}, \mathcal{A}^{\mathsf{app}}, X)$;
12 **return** $\mathcal{A}^{\mathsf{app}}$.

Consider the algorithm $\mathsf{ApproxAlg}$ that computes all $\mathbf{L}^a_{\subseteq}$-certain membership assertions for a given *DL-Lite*$_{core}$-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$. As we will show later in this section, the KB $\mathcal{T} \cup \mathsf{ApproxAlg}(\mathcal{E})$ is a minimal sound *DL-Lite*$_{core}$-approximation of $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$. $\mathsf{ApproxAlg}$ uses the algorithm $\mathsf{Weeding}$ (which was introduced in [16]) as a subroutine. Intuitively, $\mathsf{Weeding}$ works as follows: it takes as input a *DL-Lite*$_{core}$ KB $\mathcal{T} \cup \mathcal{A}$ and a set of ABox assertions $\mathcal{D}$ to be deleted, and returns as output an ABox $\mathcal{A}^w$ such that $\mathcal{A}^w \not\models_{\mathcal{T}} \mathcal{D}$. It starts with $\mathcal{A}^w = \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ (Line 1), and then for each MA $B_1(c)$ in $\mathcal{A}$, the algorithm deletes $B_1(c)$ all the assertions of $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ that $\mathcal{T}$-entail $B_1(c)$ (Lines 2-10). Intuitively, $\mathsf{ApproxAlg}$ works as follows: it takes as input a simple *DL-Lite*$_{core}$-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ and returns a *DL-Lite*$_{core}$ ABox $\mathcal{A}^{\mathsf{app}}$ such that $\mathcal{T} \cup \mathcal{A}$ is a minimal sound approximation of $\mathcal{K} \diamond_S \mathcal{N}$. First, it computes $\mathsf{AtAlg}(\mathcal{E})$ (Line 1) since the assertion which $\mathcal{T}$-contradicts $\mathcal{N}$ are not to be in the result. Second, it computes the positive MAs of $\mathsf{AtAlg}(\mathcal{E})$ which are not certain (Lines 2-7). Then, it computes the negative MAs of $\mathsf{AtAlg}(\mathcal{E})$ which are not certain (Lines 8-10). Finally, it deletes the uncertain MAs from $\mathsf{AtAlg}(\mathcal{E})$ by means of $\mathsf{Weeding}$ algorithm and adds $\mathcal{N}$ to the result (Line 11). Consider an example that illustrates how the algorithm works.

Example 3.14. Continuing with Example 3.9, we compute the approximation of $\mathcal{K}_2 \diamond_S \mathcal{N}_1$ with $S = \mathbf{L}^a_{\subseteq}$. First, we compute the necessary components $\mathsf{cl}_{\mathcal{T}}(\mathcal{A}_2)$, $\mathsf{AtAlg}(\mathcal{K}_2, \mathcal{N}_1)$, and X: we start with $X = \emptyset$, and

$$\mathsf{cl}_{\mathcal{T}}(\mathcal{A}_2) = \mathcal{A}_2 \cup \{\neg Priest(john),\ \neg\exists HasHusb^-(pedro),\ \neg\exists HasHusb^-(ivan)\},$$

$$\mathsf{AtAlg}(\mathcal{K}_2, \mathcal{N}_1) = \{Priest(pedro),\ Priest(ivan), \neg\exists HasHusb^-(pedro),\ \neg\exists HasHusb^-(ivan)\}.$$

Second, we re-compute X as in Lines 2-6: $X := X \cup \emptyset$. Third, we re-compute X as in Line 7: $X := X \cup \{Priest(pedro), Priest(ivan)\}$. Then, we delete from $\mathcal{A}^{\mathsf{app}}$ MAs $\neg\exists HasHusb(pedro)$ and $\neg\exists HasHusb(ivan)$ as in Lines 8-10; one can check that these MAs are not certain; indeed, $\mathcal{J}_1 \not\models \neg\exists HasHusb(pedro)$ and $\mathcal{J}_2 \not\models \neg\exists HasHusb(ivan)$. Finally, the $\mathsf{Weeding}(\mathcal{T}, \mathcal{A}^{\mathsf{app}}, X)$ algorithm returns $\emptyset$. Therefore, $\mathcal{A}^{\mathsf{app}} = \mathcal{N} = \{Priest(john)\}$.

To sum up, as soon as a husband *john*, who is married to some unknown individual decides to become a priest, the algorithm that computes minimal sound approximation forces us to delete all the priests and all the wives from the old knowledge. The reason is that we do not know who of the wives from the old knowledge were married to *john* and who are their new husbands: either some of the former priest, or even no one. To account for this uncertainty, the atoms about wives and priests should be erased from the old KB. Thus, minimal sound approximation $\mathcal{K}^{\mathsf{app}}$ may erase a lot of old knowledge and the approximation result may be quite unexpected from the user point of view. ∎

Before proceeding to the formal proof of the algorithm's correctness, consider the following two lemmas. They characterize positive and negative $\mathbf{L}^a_\subseteq$-certain membership assertions. The first lemma shows that positive $\mathbf{L}^a_\subseteq$-certain MAs are characterized by prototypes of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$.

Lemma 3.24. *Let $(\mathcal{K}, \mathcal{N})$ be a simple DL-Lite$_{core}$-evolution setting, g a positive membership assertion, and $\mathfrak{J}$ a prototypal set for $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$. Then, g is $\mathbf{L}^a_\subseteq$-certain for $(\mathcal{K}, \mathcal{N})$ if and only if $\mathcal{J} \models g$ for every $\mathcal{J} \in \mathfrak{J}$.*

Proof. Let $S = \mathbf{L}^a_\subseteq$, $\mathcal{K}'$ be an $\mathcal{S}$-evolution for $(\mathcal{K}, \mathcal{N})$, and $\mathfrak{J}$ a prototypal set for $\mathcal{K} \diamond_S \mathcal{N}$. The "only-if" direction is trivial. Indeed, if a positive MA g is $\mathbf{L}^a_\subseteq$-certain for $(\mathcal{K}, \mathcal{N})$, then $\mathcal{K}' \models g$ and consequently, by the definition of S-evolution, $\mathcal{K} \diamond_S \mathcal{N} \models g$. It remains to observe that $\mathfrak{J} \subseteq \mathcal{K} \diamond_S \mathcal{N}$ to conclude the proof.

We now show the "if" direction. Suppose that g is a positive MA such that $\mathcal{J} \models g$ for every $\mathcal{J} \in \mathfrak{J}$. Let $\mathcal{J}_0$ be in $\mathcal{K} \diamond_S \mathcal{N}$. Consider a prototype $\mathcal{J}_0'$ in $\mathfrak{J}$ for which there exists a homomorphism h such that $h : \mathcal{J}_0' \hookrightarrow \mathcal{J}_0$. Since $\mathcal{J}_0' \models g$, we have three possibilities:

(i) If g is of the form $A(b)$ and $b \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$, then $A(b) \in \mathcal{J}_0'$ and hence $A(h(b)) = A(b) \in \mathcal{J}_0$.

(ii) If g is of the form $R(b, c)$ and $b, c \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$, then, analogously to the previous case, $R(b, c) \in \mathcal{J}_0$.

(iii) If g is of the form $\exists R(b)$ and $b \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$, then there exists an element $\alpha \in \Delta$ such that $R(b, \alpha) \in \mathcal{J}_0'$, and therefore $R(h(b), h(\alpha)) = R(b, h(\alpha)) \in \mathcal{J}_0$.

In all the cases, we have that $\mathcal{J}_0 \models g$, which concludes the proof. □

The next lemma shows that negative $\mathbf{L}^a_\subseteq$-certain MAs for $(\mathcal{K}, \mathcal{N})$ are characterized by the elements of $\mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ or $\mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$.

Lemma 3.25. *Let $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a simple DL-Lite$_{core}$-evolution setting, and g a negative membership assertion. Then g is $\mathbf{L}^a_\subseteq$-certain for $(\mathcal{N}, \mathcal{K})$ if and only if*

$$g \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \cup \left(\mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \setminus \bigcup_{R \in \mathsf{TR}} \ \bigcup_{A(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)} \{\neg\exists R^-(c)\} \right).$$

Proof. Let $S = \mathbf{L}^a_\subseteq$.

First we show the "if" direction. Suppose that $g \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \cup Y$, where $Y = \mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \setminus \bigcup_{R \in \mathsf{TR}} \bigcup_{A(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)} \{\neg\exists R^-(c)\}$. If $g \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ then, clearly, by the definition of $\mathcal{K} \diamond_S \mathcal{N}$, g is certain. Suppose that

$$g \in Y \setminus \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \text{ and consequently } g \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N}). \tag{25}$$

Assume g is not certain, that is, there is a model $\mathcal{J}_0 \in \mathcal{K} \diamond_S \mathcal{N}$ such that $\mathcal{J}_0 \models \neg g$. Let $\mathcal{I}_0$ be a model of $\mathcal{K}$ such that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. We now exhibit a model $\mathcal{J}_0'$ such that $\mathcal{I}_0 \ominus \mathcal{J}_0' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$ which will give us a contradiction with $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. Consider the following two cases:

- g is of the form $\neg A(c)$, that is, $A(c) \in \mathcal{J}_0$. Now we show that in this case $\mathcal{N} \parallel_{\mathcal{T}} A(c)$, i.e., $\mathcal{N} \not\models_{\mathcal{T}} \neg A(c)$ and $\mathcal{N} \not\models_{\mathcal{T}} A(c)$. Indeed, *(a)* $\mathcal{N} \not\models_{\mathcal{T}} \neg A(c)$ holds since $\mathcal{J}_0 \models A(c)$ and $\mathcal{J}_0 \models \mathcal{N}$, and *(b)* $\mathcal{N} \not\models_{\mathcal{T}} A(c)$: suppose by contradiction that $\mathcal{N} \models_{\mathcal{T}} A(c)$, then $\neg A(c) \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ by the definition of AtAlg (note that in this case $\mathcal{N} \cup \{\neg A(c)\} \models_{\mathcal{T}} \bot$); on the other hand, $\neg A(c) \in Y$ by Equation (25), that is, $\neg A(c) \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ and we obtain a contradiction.
  Now consider the following interpretation: $\mathcal{J}_0' = \mathcal{J}_0 \setminus \mathsf{root}_{\mathcal{T}}(A(c))$. Due to $\mathcal{N} \parallel_{\mathcal{T}} \neg A(c)$, we have that $\mathcal{J}_0' \in \mathsf{Mod}(\mathcal{N})$. Also, $\mathcal{J}_0'$ is in $\mathsf{Mod}(\mathcal{T})$ due to Proposition 3.3. It is easy to check that, by the definition of $\mathcal{J}_0'$ and due to the restrictions of the simple evolution settings (in particular, Restriction (ii) of Definition 3.13 yields that $\mathsf{root}_{\mathcal{T}}(A(c))$ consists only of atomic MAs), the following holds: *(a)* $\mathcal{J}_0$ and $\mathcal{J}_0'$ differ only on finitely many assertions of $\mathsf{root}_{\mathcal{T}} A(c)$, and *(b)* $\mathsf{root}_{\mathcal{T}} A(c) \not\subseteq \mathcal{I}_0$ since $\neg A(c) \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$, that is, $\mathcal{K} \models \neg A(c)$, and $\mathcal{I}_0 \models \mathcal{K}$. This leads to $\mathcal{I}_0 \ominus \mathcal{J}_0' \subseteq \mathcal{I}_0 \ominus \mathcal{J}_0$. Then, $A(c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I}_0 \ominus \mathcal{J}_0')$, which means that the inclusion is strict, i.e., $\mathcal{I}_0 \ominus \mathcal{J}_0' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$. Thus, we obtain a contradiction with $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$.
- g is of the form $\neg\exists R^-(c)$, that is, there exists $\alpha \in \Delta$ such that $R(\alpha, c) \in \mathcal{J}_0$. We now show that $\mathcal{N} \parallel_{\mathcal{T}} \exists R^-(c)$. Indeed, $\mathcal{N} \not\models_{\mathcal{T}} \exists R^-(c)$ holds, otherwise, since $\mathcal{N} \cup \{\neg\exists R^-(c)\} \models_{\mathcal{T}} \bot$ (see the definition of AtAlg), it would hold that $\neg\exists R^-(c) \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$, which contradicts Equation (25); furthermore, $\mathcal{N} \not\models \neg\exists R^-(c)$ holds since $\mathcal{J}_0 \models R(\alpha, c)$ and $\mathcal{J}_0 \models \mathcal{N}$. Thus, $\mathcal{N} \parallel_{\mathcal{T}} \exists R^-(c)$.
  Observe that $\mathcal{N} \not\models \neg\exists R(\alpha)$ due to $\mathcal{J}_0 \models R(\alpha, c)$ and $\mathcal{J}_0 \models \mathcal{N}$. We are ready to define $\mathcal{J}_0'$. Consider now two following cases: $\mathcal{N} \not\models_{\mathcal{T}} \exists R(\alpha)$ and $\mathcal{N} \models_{\mathcal{T}} \exists R(\alpha)$. In the former case we have that $\mathcal{N} \parallel_{\mathcal{T}} \exists R(\alpha)$ and define $\mathcal{J}_0'$ as follows: $\mathcal{J}_0' = \mathcal{J}_0 \setminus \mathsf{root}_{\mathcal{T}}(\exists R(\alpha)) \setminus \mathsf{root}_{\mathcal{T}}(\exists R^-(c))$. Due to $\mathcal{N} \parallel_{\mathcal{T}} \exists R^-(c)$ and $\mathcal{N} \parallel_{\mathcal{T}} \exists R(\alpha)$, we have that $\mathcal{J}_0' \in \mathsf{Mod}(\mathcal{N})$. Due to Proposition 3.3 we have that $\mathcal{J}_0'$ is in $\mathsf{Mod}(\mathcal{T})$. Thus, taking into account that $(\mathsf{root}_{\mathcal{T}}(\exists R(\alpha)) \cup \mathsf{root}_{\mathcal{T}}(\exists R^-(c))) \setminus \{R(\alpha, c)\}$ consists of unary atoms only (this holds due to Restriction (ii) of Definition 3.13), we obtain that $\mathcal{I}_0 \ominus \mathcal{J}_0' \subseteq \mathcal{I}_0 \ominus \mathcal{J}_0$ and $R(\alpha, c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I}_0 \ominus \mathcal{J}_0')$, which yields $\mathcal{I}_0 \ominus \mathcal{J}_0' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$ and a contradiction with $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. In the latter case, when $\mathcal{N} \models_{\mathcal{T}} \exists R(\alpha)$, we define $\mathcal{J}_0'$ as follows: $\mathcal{J}_0' = \mathcal{J}_0 \setminus \mathsf{root}_{\mathcal{T}}(\exists R^-(c))$. Due to Restriction (iii) of Definition 3.13, it holds that $\mathcal{N} \models_{\mathcal{T}} R(\alpha, d)$ for some d. Note that $d \neq c$ since $\mathcal{N} \not\models_{\mathcal{T}} \exists R^-(c)$, and therefore $\mathcal{J}_0'$ is a model of $\mathcal{N}$ (as in the previous case, $\mathsf{root}_{\mathcal{T}}(\exists R^-(c)) \setminus \{R(\alpha, c)\}$ consists of only unary atoms). By Proposition 3.3, we have that $\mathcal{J}_0'$ is in $\mathsf{Mod}(\mathcal{T})$. As in the previous case, we obtain $\mathcal{I}_0 \ominus \mathcal{J}_0' \subseteq \mathcal{I}_0 \ominus \mathcal{J}_0$ and $R(\alpha, c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I}_0 \ominus \mathcal{J}_0')$, which yields $\mathcal{I}_0 \ominus \mathcal{J}_0' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$ a contradiction with $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$.

Therefore, if $g \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \cup Y$, then $\mathcal{K} \diamond_S \mathcal{N} \models g$.

We show now the "only-if" direction. Suppose that $\mathcal{K} \diamond_S \mathcal{N} \models g$, but $g \notin \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \cup Y$. There are two possible cases: $g \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ or $g \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$. In the former case we have that $g \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ and $g \in \bigcup_{R \in \mathsf{TR}} \bigcup_{B(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)} \{\neg\exists R^-(c)\}$, so there exists a concept B such that $\mathcal{T} \models \exists R^- \sqsubseteq \neg B$ and $\mathcal{A} \models_{\mathcal{T}} B(c)$. Observe that the prototype $\mathcal{J}[\{B(c)\}, \langle R \rangle, \langle A \rangle]$ for some $A \in \mathtt{ISubCon}(R)$ is such that it does not satisfy $\neg\exists R^-(c)$. We obtain a contradiction with the assumption that g is certain, and therefore $\mathcal{K} \diamond_S \mathcal{N} \not\models g$. Finally, suppose that $g = \neg f \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$. First, observe that $\mathcal{N} \parallel_{\mathcal{T}} f$; indeed, *(i)* since $g \notin \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$, we conclude that $\mathcal{N} \not\models_{\mathcal{T}} \neg f$, *(ii)* since $\mathcal{K} \diamond_S \mathcal{N} \models \neg f$, we conclude that $\mathcal{N} \not\models_{\mathcal{T}} f$. Second, observe that $\mathcal{A} \parallel_{\mathcal{T}} f$; indeed, *(i)* suppose that $\mathcal{A} \models_{\mathcal{T}} f$, i.e., $f \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$, then, since $\mathcal{N} \not\models_{\mathcal{T}} \neg f$, it holds that $f \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$, which is not the case, and therefore we conclude that $\mathcal{A} \not\models_{\mathcal{T}} f$; *(ii)* similarly to the previous case, one can show that $\mathcal{A} \not\models_{\mathcal{T}} \neg f$. Recall that $\mathcal{K} \diamond_S \mathcal{N} \models \neg f$. Consider models $\mathcal{J}_0 \in \mathcal{K} \diamond_S \mathcal{N}$ and $\mathcal{I}_0 \in \mathsf{Mod}(\mathcal{K})$ such that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. Then, consider a set $Y = \mathsf{chase}_{\mathcal{T}}(f)$, where all the Δ-elements are such that they do not occur in $\mathsf{adom}(\mathcal{K} \cup \mathcal{N})$ and models $\mathcal{I}_0$ and $\mathcal{J}_0$. Using Y we define the following two models: $\mathcal{I}'_0 = \mathcal{I}_0 \cup Y$ and $\mathcal{J}'_0 = \mathcal{J}_0 \cup Y$. It is easy to check that $\mathcal{I}'_0 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{A})$ and $\mathcal{J}_0 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$. We are going to show now that $\mathcal{J}'_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}'_0, \mathcal{T}, \mathcal{N})$, which will lead to a contradiction with the fact that $\mathcal{K} \diamond_S \mathcal{N} \models g$ since $\mathcal{J}'_0 \not\models g$. Suppose that $\mathcal{J}'_0 \notin \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}'_0, \mathcal{T}, \mathcal{N})$, that is, there exists a model $\mathcal{J}''_0 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $\mathcal{I}'_0 \ominus \mathcal{J}''_0 \subsetneq \mathcal{I}'_0 \ominus \mathcal{J}'_0$. Thus, there is an atom $f' \in \mathcal{J}'_0$ such that $f' \in (\mathcal{I}'_0 \ominus \mathcal{J}'_0) \setminus (\mathcal{I}'_0 \ominus \mathcal{J}''_0)$. Note that $f' \notin Y$ since $Y \not\subseteq \mathcal{I}'_0 \ominus \mathcal{J}'_0$, and also $\mathcal{J}'' = \mathcal{J}''_0 \setminus (Y \setminus \mathcal{J}_0)$ is in $\mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$. These two observations lead to the fact that $\mathcal{I}_0 \ominus \mathcal{J}'' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$ which contradicts $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. Therefore, $\mathcal{J}'_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}'_0, \mathcal{T}, \mathcal{N})$. Since $\mathcal{J}'_0 \not\models g$ we conclude that g is not certain. □

Based on the preceding lemmas, we conclude with the main result of this section:

Theorem 3.26. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a simple DL-Lite$_{core}$-evolution setting and let $\mathcal{A}^{\mathsf{app}} = \mathsf{ApproxAlg}(\mathcal{E})$. Then,*
(i) g is $\mathbf{L}^a_{\subseteq}$-certain for $\mathcal{E}$ if and only if $g \in \mathcal{A}^{\mathsf{app}}$, and
(ii) $\mathcal{K}^{\mathsf{app}} = \mathcal{T} \cup \mathcal{A}^{\mathsf{app}}$ is a minimal sound DL-Lite$_{core}$-approximation of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$.
Moreover, $\mathsf{ApproxAlg}(\mathcal{E})$ can be computed in time polynomial in $|\mathcal{E}|$.

Proof. The proof follows from Lemma 3.25, and Lemma 3.24 coupled with the correctness of the the BP procedure.

More precisely, polynomiality follows from polynomiality of AtAlg and Weeding algorithms and polynomiality of computing $\mathsf{TR}[\mathcal{T}, \mathcal{N}]$ and $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$.

Case (i): Suppose that an MA g is certain. Consider the following two cases.

- Assume that g is positive. Then, by Lemma 3.24, for every $\mathcal{J} \in \mathsf{BZP}(\mathcal{E})$ it holds that $\mathcal{J} \models g$. From BP and BZP, it is easy to see that the prototypes from $\mathsf{BZP}(\mathcal{E})$ differ only on atoms including roles from $\mathsf{TR}[\mathcal{T}, \mathcal{N}]$, MAs from $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$, and those atoms that entail them. Clearly, those atoms are uncertain. The atoms on which the prototypes differ are added to X by the algorithm ApproxAlg and deleted at Line 11 by means of Weeding algorithm.
- Assume that g is negative. Then, by Lemma 3.25, it belongs to $\mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ or $\mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \setminus Z$, where $Z = \bigcup_{R \in \mathsf{TR}} \bigcup_{A(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)} \{\neg \exists R^-(c)\}$. In the former case, it holds that $\mathcal{A}^{\mathsf{app}} \models_{\mathcal{T}} g$ since $\mathcal{N} \subseteq \mathcal{A}^{\mathsf{app}}$. In the latter case, it again holds that $\mathcal{A}^{\mathsf{app}} \models_{\mathcal{T}} g$ since the algorithm deletes negative MAs from $\mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ only in Line 9, and those MAs are from Z.

Case (ii): Suppose that $\mathcal{T} \cup \mathcal{A}^{\mathsf{app}}$ is not minimal sound approximation, that is, there is an MA g' such that $\mathcal{T} \cup \mathcal{A}^{\mathsf{app}} \cup \{g'\}$ is a sound approximation. Note that g' is not certain since $\mathcal{T} \cup \mathcal{A}^{\mathsf{app}}$ entails all the certain MAs. Thus, by Definition 3.19, there is a model $\mathcal{J}' \in \mathcal{K} \diamond_S \mathcal{N}$ such that $\mathcal{J}' \not\models g'$. Clearly, $\mathcal{J}' \notin \mathsf{Mod}(\mathcal{T} \cup \mathcal{A}^{\mathsf{app}} \cup \{g'\})$, that is, $\mathcal{T} \cup \mathcal{A}^{\mathsf{app}} \cup \{g'\}$ is not a sound approximation. The obtained contradiction concludes the proof. □

Summary on Section 3.5 For *DL-Lite$_{core}$*-evolution settings, $\mathbf{L}^a_{\subseteq}$-evolution can be efficiently *DL-Lite$_{core}$*-approximated and we presented an algorithm ApproxAlg which can be used to compute these approximations.

3.6 Summary on Model-Based Evolution

We summarize here on how one can use the results established in this section to do ABox evolution of *DL-Lite$_{core}$* KBs under model-based semantics in practice. Given a *DL-Lite$_{core}$* evolution setting $(\mathcal{K}, \mathcal{N})$, one can first check (in polynomial time) whether $\mathcal{K}$ is in *DL-Lite^{pr}*. If this is the case, then one can compute in polynomial time S-evolution for $\mathcal{E}$, where S is any of atom-based semantics $\mathbf{G}^a_{\subseteq}$, $\mathbf{L}^a_{\subseteq}$, $\mathbf{G}^a_{\#}$, and $\mathbf{L}^a_{\#}$, or global symbol-based semantics $\mathbf{G}^s_{\subseteq}$ and $\mathbf{G}^s_{\#}$, using the techniques of Theorems 3.5 or 3.7, respectively. One can also compute a minimal sound *DL-Lite^{pr}*-approximation of S-evolution under the remaining two local symbol-based semantics $\mathbf{L}^s_{\subseteq}$ and $\mathbf{L}^s_{\#}$ using the techniques of Theorem 3.9. The choice of evolution semantics for *DL-Lite^{pr}* is up to the user, while we believe that atom-based semantics behave more intuitively. For the case when $\mathcal{K}$ is *not* in *DL-Lite*^{pr}, the set of evolved models $\mathcal{K} \diamond_S \mathcal{N}$ is in general not axiomatizable in *DL-Lite*$_{core}$ for S being any of the eight MBAs [16, 18]. At the same time, if $S = \mathbf{L}^a_{\subseteq}$, then one can compute in polynomial time a minimal sound *DL-Lite*$_{core}$-approximation of $\mathcal{K} \diamond_S \mathcal{N}$ using the techniques of Theorem 3.26.

4 Formula-Based Evolution

The formula-based approach to evolution relies on the notion of deductive closure of KBs. Given an evolution setting $\mathcal{E}$, the *bold semantics* (**BS**) [16] is defined as follows:

Definition 4.1 (**BS** Semantics)**.** Let $\mathcal{D}$ be a DL and $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, a $\mathcal{D}$-evolution setting and $\mathcal{M}(\mathcal{E})$ the class of maximal subsets $\mathcal{S}$ of $\mathsf{cl}_{\mathcal{D}}(\mathcal{K})$ s.t. $\mathcal{S} \models \mathcal{N}$. Then, $\mathcal{K}' \in \mathcal{D}$ is **BS**-*evolution*, for $\mathcal{E}$, denoted $\mathcal{K} \diamond_{\mathbf{BS}} \mathcal{N}$, if there exists $\mathcal{S} \in \mathcal{M}(\mathcal{E})$ such that $\mathcal{K}' \equiv \mathcal{S}$.

Example 4.1. Consider the KB $\mathcal{K}$ that consists of TBox $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq C\}$ and an empty ABox. Let $\mathcal{N} = \{A \sqsubseteq \neg C\}$. Note that evolution of $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ is not ABox evolution, instead it is a TBox evolution. There are two possible **BS**-evolutions for $\mathcal{E}$: $\mathcal{K}'_1 = \{A \sqsubseteq B\} \cup \mathcal{N}$ and $\mathcal{K}'_2 = \{B \sqsubseteq C\} \cup \mathcal{N}$.

Another example is when *john* is becoming a priest, as the following evolution setting $\mathcal{E}_0$ shows:

$$\begin{aligned}
\mathcal{T}_0 &= \{\mathit{HasHusb} \sqsubseteq \neg \mathit{Priest},\ \exists \mathit{HasHusb}^- \sqsubseteq \mathit{Woman}\},\\
\mathcal{A}_0 &= \{\mathit{HasHusb}(\mathit{john}, \mathit{mary}),\ \mathit{Priest}(\mathit{pedro})\},\\
\mathcal{N}_0 &= \{\mathit{Priest}(\mathit{john})\}.
\end{aligned}$$

There is a unique **BS**-evolutions for $\mathcal{E}_0$, i.e.,

$$\mathcal{K}'_0 = \mathcal{T}_0 \cup \mathcal{A}_0 \setminus \{\mathit{HasHusb}(\mathit{john}, \mathit{mary})\} \cup \{\mathit{Priest}(\mathit{john}),\ \exists \mathit{HasHusb}^-(\mathit{mary})\}. \qquad \blacksquare$$

Note that under Bold semantics $\mathcal{K}'$ is not unique in general (even modulo equivalence). There have been several proposals for combining all elements of $\mathcal{M}(\mathcal{C})$ into a single set of formulas [16, 23, 50]. Under *Cross-Product* (**CP**) semantics, evolution is equivalent to the "disjunction" of all relevant maximal subsets of the closure, whereas under *When In Doubt Throw It Out* (**WIDTIO**) semantics, one takes the "intersection". Formally:

Definition 4.2 (**CP** and **WIDTIO** Semantics)**.** Let $\mathcal{D}$ be a DL and $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a $\mathcal{D}$-evoliution setting. Then, $\mathcal{K}' \in \mathcal{D}$ is a **CP**-*evolution* for $\mathcal{E}$, denoted $\mathcal{K} \diamond_{\mathbf{BS}} \mathcal{N}$, if

$$\mathcal{K}' \equiv \{ \bigvee_{\mathcal{S} \in \mathcal{M}(\mathcal{E})} (\bigwedge_{\beta \in \mathcal{S}} \beta)\},$$

and $\mathcal{K}' \in \mathcal{D}$ is a **WIDTIO**-*evolution* for $\mathcal{E}$, denoted $\mathcal{K} \diamond_{\mathbf{WIDTIO}} \mathcal{N}$, if

$$\mathcal{K}' \equiv \bigcap_{\mathcal{S} \in \mathcal{M}(\mathcal{E})} \mathcal{S}.$$

Example 4.2. Continuing with TBox evolution of Example 4.1, $\mathcal{K} \diamond_{\mathbf{CP}} \mathcal{N} = \{(A \sqsubseteq B) \vee B \sqsubseteq C\}$, while $\mathcal{K} \diamond_{\mathbf{WIDTIO}} \mathcal{N} = \emptyset$.

For the case of John's priesthood, we have that both **CP**-evolution **WIDTIO**-evolution coincide and equal to $\mathcal{K}'_0$. $\blacksquare$

Expressibility of optimal evolutions can now be defined in the obvious way.

Definition 4.3 (Expressibility for FBAs)**.** Let $\mathcal{D}$ be a DL and let $\mathbf{S} \in \{\mathbf{BS}, \mathbf{CP}, \mathbf{WIDTIO}\}$ be an FBAs evolution semantics. We say $\mathcal{D}$ is *closed under* $\mathbf{S}$-*evoliution* (or $\mathbf{S}$-evolution is *expressible* in $\mathcal{D}$) if for each $\mathcal{D}$-evolution setting $\mathcal{E}$, an $\mathbf{S}$-evolution $\mathcal{K}' \in \mathcal{D}$ exists.

Intuitively, **CP** has the advantage of not "losing information"; however, **CP**-evolutions can be exponentially larger than the original ontology, even if its closure is a finite set. In addition, even if we consider a *DL-Lite*-evoliution setting $\mathcal{E}$, the corresponding **CP**-evolution for $\mathcal{E}$ may not be expressible in *DL-Lite*, since a language with disjunction may be required. In contrast, **WIDTIO**-evolutions for *DL-Lite* are always expressible, but important information may be lost. Furthermore, it is generally difficult to decide whether a formula belongs to **WIDTIO**-evolution $\mathcal{K}'$. This problem is already difficult for TBox evolution (i.e. evolution when $\mathcal{N}$ is a TBox) in the simplest variant of *DL-Lite*: in [16] it was shown that for the problem of update, deciding whether $A \sqsubseteq B \in \mathcal{K}'$ is coNP-complete. Thus, both **CP** and **WIDTIO** semantics are somewhat problematic, even for languages such as *DL-Lite*, where the deductive closure is always a finite set.

In contrast, **BS** semantics is well-behaved for *DL-Lite*: optimal evolutions always exist; furthermore, in the case of ABox evolution they are also unique and computable in polynomial time [16]. We will discuss this in more details in this section.

4.1 Properties of Formula-Based Semantics

We start with an observation that for *DL-Lite*$_{core}$-evolution settings **BS**-evolution is always unique. Before proceeding to a formal proof of this statement consider the following property of *DL-Lite*$_{core}$:

Lemma 4.1. *Let $\mathcal{T} \cup \mathcal{A}$ be a DL-Lite$_{core}$ KB. If $\mathcal{T} \cup \mathcal{A}$ is unsatisfiable, then there is a subset $\mathcal{A}_0 \subseteq \mathcal{A}$ with at most two elements, such that $\mathcal{T} \cup \mathcal{A}_0$ is unsatisfiable.*

Proof. Observe that $\mathcal{K}$ is satisfiable if and only if the Skolemized version of $\mathcal{K}$, say $\mathcal{K}_s$, is satisfiable. The set $\mathcal{K}_s$ is a set of Horn clauses. It is known that, if such a set is unsatisfiable, there is a subset with exactly one negative clause that is unsatisfiable. The only negative clauses in $\mathcal{K}_s$ are denials that stem from disjointness axioms of the form $A \sqsubseteq \neg B$. Suppose that $\mathcal{K}_s$ is unsatisfiable and that $\hat{\mathcal{K}}_s$ is an unsatisfiable subset with at most one negative clause stemming from $B \sqsubseteq \neg C$. Then, because of the form of the assertions in $\mathcal{T}$, there are membership assertions $B_0(a)$, $C_0(a)$ in $\hat{\mathcal{K}}_s$, for concepts B_0 and C_0 such that $\mathcal{T} \models B_0 \sqsubseteq B$ and $\mathcal{T} \models C_0 \sqsubseteq C$. □

We are ready to show uniqueness of *DL-Lite*$_{core}$-evolutions.

Theorem 4.2. *For every DL-Lite$_{core}$-evolution setting $\mathcal{E}$ there is the unique up to logical equivalence* **BS**-*evolution for $\mathcal{E}$.*

Proof. Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a *DL-Lite*$_{core}$-evolution setting where $\mathcal{K} = \mathcal{T} \cup \mathcal{N}$. By the definition of **BS**-evolution the result of evolution $\mathcal{K} \diamond_{\mathbf{BS}} \mathcal{N}$ is a *DL-Lite*$_{core}$ KB $\mathcal{T} \cup \mathcal{A}' \cup \mathcal{N}$, where is $\mathcal{A}'$ is a maximal subset of $\mathcal{A}$ which is $\mathcal{T}$-satisfiable with $\mathcal{N}$. So, the uniqueness of the result of evolution can be provided by uniqueness of $\mathcal{A}'$.

We will show that the subset $\mathcal{A}'$ of $\mathcal{A}$ defined as follows

$$\mathcal{A}' = \{F \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \mid F \text{ is } \mathcal{T}\text{-satisfiable with } \mathcal{N}\}.$$

is $\mathcal{T}$-satisfiable with $\mathcal{N}$, maximal, and unique.

To prove that $\mathcal{A}'$ is $\mathcal{T}$-satisfiable with $\mathcal{N}$, by Lemma 4.1, we only have to show that for each $X \subseteq \mathcal{A}' \cup \mathcal{N}$ of at most two elements, $\mathcal{T} \cup X$ is satisfiable. If $X = \{F\}$, then either $F \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ or $F \in \mathcal{N}$, and since both $\mathcal{T} \cup \mathcal{A}$ and $\mathcal{T} \cup \mathcal{N}$ are satisfiable, also $\mathcal{T} \cup X$ is satisfiable. Let us consider the case where $X = \{F_1, F_2\}$. If $X \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ or $X \subseteq \mathcal{N}$, we can argue as before. Instead, if $F_1 \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ and $F_2 \in \mathcal{N}$, then $\mathcal{T} \cup X$ is satisfiable because by definition of $\mathcal{A}'$ we have that F_1 is $\mathcal{T}$-satisfiable with $\mathcal{N}$.

To see that $\mathcal{A}'$ is also maximal, assume that there is a subset $\hat{\mathcal{A}}$ of $\mathcal{A}$ such that $\mathcal{A}' \subset \hat{\mathcal{A}} \subset \mathcal{A}$ and that also $\mathcal{T} \cup \hat{\mathcal{A}}$ is satisfiable. Then, for every $F \in \hat{\mathcal{A}}$, we have that F is $\mathcal{T}$-satisfiable with $\mathcal{N}$, hence $F \in \mathcal{A}'$ by definition. Thus, $\hat{\mathcal{A}} \subseteq \mathcal{A}'$.

Algorithm 6: $\mathsf{BSAlg}(\mathcal{E})$

INPUT : *DL-Lite*$_{core}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$
OUTPUT : The maximal set $\mathcal{A}' \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ of ABox assertions that is $\mathcal{T}$-satisfiable with $\mathcal{N}$

1 $\mathcal{A}' := \mathcal{N}$; $\mathcal{S} := \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$;
2 **repeat**
3   **choose some** $\phi \in \mathcal{S}$; $\mathcal{S} := \mathcal{S} \setminus \{\phi\}$;
4   **if** $\{\phi\} \cup \mathcal{A}'$ *is* $\mathcal{T}$*-consistent* **then** $\mathcal{A}' := \mathcal{A}' \cup \{\phi\}$
5 **until** $\mathcal{S} = \emptyset$;
6 **return** $\mathcal{A}'$.

Finally, to see that $\mathcal{A}'$ is unique, suppose that it is not. So, there exists another maximal subset $\mathcal{A}'_1$ of $\mathcal{A}$ which is $\mathcal{T}$-satisfiable with $\mathcal{N}$. Consider the subset $\mathcal{A}'' = \mathcal{A}'_1 \setminus \mathcal{A}'$. Arguing as in previous paragraph with $\hat{\mathcal{A}} = \mathcal{A}' \cup \mathcal{A}''$, we can show that $\mathcal{A}'' = \emptyset$ which proves the uniqueness of $\mathcal{A}'$. □

As a corollary from the theorem we conclude that formula-based semantics considered above coincide.

Corollary 4.3. *For DL-Lite*$_{core}$*-evolution settings* **BS**, **WIDTIO** *and* **CP** *semantics coincide.*

We now present the algorithm BSAlg, see Algorithm 6, that computes $\mathcal{K} \diamond_{\mathbf{BS}} \mathcal{N}$ in polynomial time. Give a *DL-Lite*$_{core}$-evolution setting $(\mathcal{K}, \mathcal{N})$ where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, BSAlg returns an ABox $\mathcal{A}'$ such that $\mathcal{T} \cup \mathcal{A}'$ is **BS**-evolution for $\mathcal{E}$. The algorithm loops as many times as there are assertions in $\mathsf{cl}(\mathcal{K})$. The number of such assertions is at most quadratic in the number of constants, atomic concepts, and roles. The crucial step is the check for consistency, which is performed once per loop. This test is polynomial in the size of the input and therefore the entire runtime of the algorithm is polynomial.

Theorem 4.4. *Let* $\mathcal{E}$ *be a DL-Lite*$_{core}$*-evolution setting. Then,* $\mathcal{T} \cup \mathsf{BSAlg}(\mathcal{E})$ *is a* **BS***-evolution for* $\mathcal{E}$ *and it can be computed in time polynomial in* $\mathcal{E}$*.*

Proof. First, we show the correctness of the algorithm. Note that by construction, the output $\mathcal{A}' = \mathsf{BSAlg}(\mathcal{E})$ is $\mathcal{T}$-consistent with $\mathcal{N}$. Then suppose that $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$ is not a maximal subset of $\mathcal{K}$, that is, there exists an ABox assertion F such that $\mathcal{K}' \cup \mathcal{N} \cup \{F\}$ is satisfiable. Let $\mathcal{A}'_i$ be a set built by the algorithm BSAlg at Step (i) (the clause **repeat-until** in Lines 2-5 goes over all the assertions of $\mathcal{A}$, we designate each pass as step). Since $F \notin \mathcal{K}'$ it was dropped at some Step (j) in Line 4, i.e., $\mathcal{A}'_{j-1} \cup \{F\}$ is not $\mathcal{T}$-satisfiable. Hence, $\mathcal{A}' \cup \{F\} \models_{\mathcal{T}} \bot$ due to $\mathcal{A}'_{j-1} \subseteq \mathcal{A}'$. Thus, $\mathcal{A}'$ is a maximal subset of $\mathcal{A}$ $\mathcal{T}$-satisfiable with $\mathcal{N}$.

Now we prove the polynomiality of the algorithm. Since computing closure (Line 1) takes polynomial time, the crucial part of the algorithm is checking satisfiability at each step. Polynomiality of this check follows from the fact that each consistency conflict involves two membership assertions (Lemma 4.1). If ϕ is a concept membership assertion $C(a)$ for some concept C and some constant a, then we should check whether it will cause $\mathcal{T}$-inconsistency with $\mathcal{A}'_{i-1} \cup \mathcal{N}$, that is, to find whether $\mathcal{T}$ includes an assertion $C \sqsubseteq \neg C'$ (or $C' \sqsubseteq \neg C$), and if it does, to check whether $\mathcal{A}'_{i-1} \cup \mathcal{N}$ includes an assertion $C'(a)$. It is easy to see that the check can be done in polynomial time. Observe that *DL-Lite*$_{core}$ allows for disjointness of concepts only, thus if ϕ is a role membership assertion $R(a, b)$ for some role R and some constants a and b, then $\{R(a, b)\} \cup \mathcal{A}'_{i-1}$ is always $\mathcal{T}$-consistent. □

4.2 Making Bold Semantics More Careful

We start with an example illustrating drawbacks of **BS**.

Example 4.3. Coming back to $\mathcal{E}_0$ of Example 4.1, observe that as the result of evolution the only assertion to be dropped from $\mathcal{K}_0$ is that John is the husband of Mary, i.e., $\mathit{HasHusb}(\mathit{john}, \mathit{mary})$, while we had to keep the assertions $\exists \mathit{HasHusb}^-(\mathit{mary})$. This implies that Mary *still has a husband* who is not John, despite the divorce with John, that is, $\mathcal{K}_0 \diamond_{\mathbf{BS}} \mathcal{N} \models \phi$, where $\phi = \exists x(\mathit{HasHusb}(\mathit{mary}, x) \wedge (x \neq \mathit{john}))$. The only option that **BS** offers to Mary is to find another husband immediately after the divorce. It does not consider it an option for her to become single. This phenomenon is due to the fact that $\mathcal{N}_0$ has not explicitly said that Mary got divorced, it only said that John is not her husband anymore. We are interested in a semantics that allows for both possibilities. Note that the entailment $\mathcal{K}_0 \diamond_{\mathbf{BS}} \mathcal{N} \models \phi$ is *unexpected* in the sense that neither $\mathcal{K}_0$ nor $\mathcal{N}$ entail ϕ, that is, $\mathcal{K}_0 \not\models \phi$ and $\mathcal{N} \not\models \phi$ hold. ∎

As we see from the example, the situation when the result of evolution entails *unexpected information*, that is, information coming neither from the original KB, nor from the new knowledge, may be counterintuitive. In our example, the unexpected information is the formula $\exists x(\mathit{HasHusb}(\mathit{mary}, x) \wedge (x \neq \mathit{john}))$, which has a specific form: it restricts the possible values in the second component of the role *HasHusb*. Our next semantics prohibits these role restrictions from being unexpectedly entailed from the result of evolution.

Definition 4.4 (Role Constraining Formulae and Careful Sets)**.** We say that a formula is *role-constraining*, or an *RCF* for short, if it is of the form:

$$\exists x(R(a, x) \wedge (x \neq c_1) \wedge \cdots \wedge (x \neq c_n)),$$

where a and all c_i are constants.

Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a *DL-Lite*$_{core}$ evolution setting. A subset $\mathcal{A}_1 \subseteq \mathcal{A}$ is *careful for* $\mathcal{E}$ if for every RCF φ, whenever $\mathcal{A}_1 \cup \mathcal{N} \models_{\mathcal{T}} \varphi$ holds, either $\mathcal{A}_1 \models_{\mathcal{T}} \varphi$ or $\mathcal{N} \models_{\mathcal{T}} \varphi$ holds.

Using the careful sets we will define careful semantics as follows:

Definition 4.5. Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a *DL-Lite*$_{core}$-evolution setting and let $\mathcal{A}^c$ be careful for $\mathcal{E}$. Then $\mathcal{K}' = \mathcal{T} \cup (\mathcal{N} \cup \mathcal{A}^c)$ is a **CS**-*evolution* for $\mathcal{E}$, denoted $\mathcal{K} \diamond_{\mathbf{CS}} \mathcal{N}$.

We will prove that **CS**-evolution is uniquely defined form every *DL-Lite*$_{core}$-evolution setting. To this effect we will show the following several properties. We will use the following notation: $\mathcal{A} \models_{\mathcal{T}} \neg\exists R^-(c)$ denotes the fact that for every $\mathcal{I} \models \mathcal{T} \cup \mathcal{A}$ it holds that $c \notin (\exists R^-)^{\mathcal{I}}$.

The first lemma characterizes role-constraining formulae.

Lemma 4.5. *Let $\mathcal{T} \cup \mathcal{A}$ be a DL-Lite$_{core}$ KB and $\exists x(R(a, x) \wedge \bigwedge_i^n x \neq c_i)$ be a role-constraining formula. Then,*

$$\mathcal{T} \cup \mathcal{A} \models \exists x(R(a, x) \wedge \bigwedge_i^n x \neq c_i)$$

if an only if at least one of the following holds

(i) $\mathcal{A} \models_{\mathcal{T}} R(a, b)$ *for some* b *and* $b \neq c_i$ *for* $c_i \in \{c_1, \ldots, c_n\}$*, or*

(ii) $\mathcal{A} \models_{\mathcal{T}} \exists R(a)$ *and* $\mathcal{A} \models_{\mathcal{T}} \neg\exists R^-(c_i)$ *for* $i \in \{1, \ldots, n\}$.

Proof. Let $\varphi = \exists x(R(a, x) \wedge \bigwedge_i^n x \neq c_i)$. We start with the "if" direction. It clearly holds that:

$$R(a, b) \models \varphi \quad \text{and} \quad \{\exists R(a), \neg\exists R^-(c_1), \ldots, \neg\exists R^-(c_n)\} \models \varphi.$$

Therefore, if either Condition (i) or (ii) of the current lemma holds, then $\mathcal{T} \cup \mathcal{A} \models \varphi$.

Now we show the "only if direction". Assume that $\mathcal{T} \cup \mathcal{A} \models \varphi$ holds but neither Condition (i), nor (ii) does.

- The fact that Condition (i) does not hold implies that for every constant b we have $\mathcal{A} \not\models_{\mathcal{T}} R(a, b)$. Thus, $\mathcal{A} \not\models_{\mathcal{T}} \phi$ which yields a contradiction.

- The fact that Condition (ii) does not hold implies that either $\mathcal{A} \not\models_{\mathcal{T}} \exists R(a)$ or $\mathcal{A} \not\models_{\mathcal{T}} \neg\exists R^-(c_i)$ for some $1 \leq i \leq n$. The first option trivially leads to a contradiction with $\mathcal{A} \models_{\mathcal{T}} \phi$, thus, only the second option is possible. Thus, assume $\mathcal{A} \not\models_{\mathcal{T}} \neg\exists R^-(c_1)$ (the case when $\mathcal{A} \not\models_{\mathcal{T}} \neg\exists R^-(c_i)$ for some $c_i \in \{c_2, \ldots, c_n\}$ is analogous). Since $\mathcal{A} \models_{\mathcal{T}} \exists R(a)$, then in every model of $\mathcal{T} \cup \mathcal{A}$, there is at least one R-successor of the constant a. Since $\mathcal{A} \not\models_{\mathcal{T}} \neg\exists R^-(c_1)$, there is a model $\mathcal{I}_0$ of $\mathcal{T} \cup \mathcal{A}$ such that in $\mathcal{I}_0$ the only R-successor of this a is c_1. Hence, $\mathcal{I}_0 \not\models \varphi$, which contradicts our assumption that $\mathcal{T} \cup \mathcal{A} \models \varphi$.

Thus, if $\mathcal{T} \cup \mathcal{A} \models \varphi$, then at least one of Conditions (i) or (ii) holds. □

The following proposition essentially says that binary ABox assertions can be derived from single atoms only.

Proposition 4.6. *Let $\mathcal{T}$ be a TBox and $\mathcal{A}_1$, $\mathcal{A}_2$ two ABoxes satisfiable with $\mathcal{T}$. Let $R(a, b)$ be an ABox assertion such that $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \{R(a, b)\}$ is a $\mathcal{T}$-satisfiable. Then, $\mathcal{A}_1 \cup \mathcal{A}_2 \models_{\mathcal{T}} R(a, b)$ if and only if $\mathcal{A}_1 \models_{\mathcal{T}} R(a, b)$ or $\mathcal{A}_2 \models_{\mathcal{T}} R(a, b)$.*

Proof. Follows immediately from the definition of *DL-Lite*$_{core}$. □

The following proposition show that a role-constraining formula with n constants can hold only when it already holds with a smaller number of constants.

Proposition 4.7. *Let $c_1, \ldots, c_n$ be constants and there is $1 \leq i \leq n$ s.t. $\mathcal{A} \not\models_{\mathcal{T}} \exists x(R(a, x) \wedge x \neq c_i)$. Then,*

$$\mathcal{A} \not\models_{\mathcal{T}} \exists x(R(a, x) \wedge \bigwedge_i^n x \neq c_i). \tag{26}$$

Proof. Let $\mathcal{T} \cup \mathcal{A} \not\models \exists x(R(a, x) \wedge x \neq c_i)$ holds. Then, there is a model $\mathcal{J}$ of $\mathcal{T} \cup \mathcal{A}$ such that $(a, c) \notin R^{\mathcal{I}}$ for every $c \neq c_i$. Now assume that Equation (26) does not hold. That is, $\mathcal{T} \cup \mathcal{A} \models \exists x(R(a, x) \wedge \bigwedge_i^n x \neq c_i)$. Then for every model $\mathcal{I}$ of $\mathcal{T} \cup \mathcal{A}$, there exists an element $c \in \Delta$ such that $c \neq c_i$ and $(a, c) \notin R^{\mathcal{I}}$, which yields a contradiction. □

Based on the previous observations we conclude a two-witness property which will allow us to analyze role-constraining formulas by first splitting them to the ones with one constant only.

Lemma 4.8 (Two Witnesses Property)**.** *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL-Lite$_{core}$-evolution setting. If there is an unexpected role-constraining formula φ such that $\mathcal{T} \cup \mathcal{A} \models \varphi$, then*

(i) there are two assertions $C_1(a)$ and $C_2(b)$ such that $\mathcal{A} \models_{\mathcal{T}} C_1(a)$ and $\mathcal{N} \models_{\mathcal{T}} C_2(b)$, and

(ii) $\{C_1(a), C_2(b)\}$ entails an unexpected role-constraining formula of the form $\exists x(R(a, x) \wedge (x \neq b))$ or $\exists x(R(b, x) \wedge (x \neq a))$ for some role name R occurring in $\mathcal{A}$ or $\mathcal{N}$.

Sketch. The proof follows from Lemma 4.5 and Propositions 4.7, and 4.6. □

Theorem 4.9. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL-Lite$_{core}$-evolution setting. Then, the set*

$$\{\mathcal{A}_0 \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \mid \mathcal{A}_0 \textit{ is careful and } \mathcal{A}_0 \cup \mathcal{N} \textit{ is } \mathcal{T}\textit{-satisfiable}\}$$

has a unique maximal element wrt set inclusion.

Proof. Without losing generality we assume that $\mathcal{A} = \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{N} = \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$. Take $\mathcal{A}' = \mathcal{K} \diamond_{\mathbf{BS}} \mathcal{N}$, i.e., a maximal satisfiable subset of $\mathcal{A}$, which exists and is unique. We now construct $\mathcal{A}_0$.

Let $\mathcal{A}_0 := \mathcal{A}'$. Assume there is a role-constraining formula

$$\varphi = \exists x(R(a, x) \wedge \bigwedge_i^n (x \neq c_i)).$$

Then, due to Lemma 4.5, this φ is entailed from some $R(a,b)$, or from $X = \{\exists R(a), \neg\exists R^-(c_1), \ldots, \neg\exists R^-(c_n)\}$. Note that this two options are not mutually excluding. In the former case, by Proposition 4.6, φ is also entailed either from $\mathcal{A}_0$, or from $\mathcal{N}$. Thus, φ is not unexpected. If the latter case holds, but the former does not holds, then one the following cases holds:

$$\textit{(i)}\ X \subseteq \mathcal{A}_0 \quad \text{or} \quad \textit{(ii)}\ X \subseteq \mathcal{N} \quad \text{or} \quad \textit{(iii)}\ X \not\subseteq \mathcal{A}_0 \text{ and } X \not\subseteq \mathcal{N}.$$

In Case (i) and (ii), φ is not unexpected. In Case (iii) it is unexpected and we modify $\mathcal{A}_0$ in order to prohibit the entailment of φ. Let $X_i = \{\exists R(a), \neg\exists R^-(c_i)\}$.

Consider two alternatives on how one should modify $\mathcal{A}_0$:

(a) If $\exists R(a)$ is in $\mathcal{N}$. Then assume that $\neg\exists R(c_{i_j})$ for $j = \{1, \ldots, k\}$ are all elements of X in $\mathcal{A}_0 \setminus \mathcal{N}$. Now set $\mathcal{A}_0 := \mathcal{A}_0 \setminus \bigcup_{j=1}^{k} \{\neg\exists R(c_{i_j})\}$. Due to Proposition 4.7 it holds that $\mathcal{A}_0 \cup \mathcal{N} \not\models \varphi$.

(b) If $\exists R(a) \in \mathcal{A}_0$ and $\exists R(a) \notin \mathcal{N}$. Then, set $\mathcal{A}_0 := \mathcal{A}_0 \setminus \{\exists R(a)\}$. Since $\mathcal{A}_0 \not\models \exists R(a)$, we obtain that $\mathcal{A}_0 \cup \mathcal{N} \not\models \varphi$.

Afterwords, delete from $\mathcal{A}_0$ all the assertions that $\mathcal{T}$-entail ones deleted in Alternatives (a) or (b). By iterating the procedure described above for every unexpected role-constraining formula (there are finitely many of them) one obtains a careful $\mathcal{A}_0$.

Observe that $\mathcal{A}_0$ constructed as above meets the maximality w.r.t. set inclusion. Indeed, if it is not the case, then, there is an assertion $C(d)$ in $\mathcal{A} \setminus \mathcal{A}_0$ such that $\mathcal{A}_0 \cup \{C(d)\}$ is careful and $\mathcal{N} \cup \mathcal{A}_0 \cup \{C(d)\}$ is satisfiable. Two cases are possible: $C(d) \in \mathcal{A}_0 \setminus \mathcal{A}'$ or $C(d) \in \mathcal{A} \setminus \mathcal{A}$,

(i) If $C(d) \in \mathcal{A}_0 \setminus \mathcal{A}'$, then, due to maximality of $\mathcal{A}'$ and Lemma 4.1, the ABox $\mathcal{N} \cup \{C(d)\}$ is $\mathcal{T}$-unsatisfiable, which yields a contradiction.

(ii) If $C(d) \in \mathcal{A}' \setminus \mathcal{A}_0$, then $C(d)$ was dropped from A_m during its modification in one of two steps. If it was dropped due to Alternative a, then $C(d) \models_{\mathcal{T}} \neg\exists R^-(d)$ and there is $\exists R(c) \in \mathcal{N} \setminus \mathcal{A}'$. Then, $\exists x(R(c,x) \wedge (x \neq d))$ is unexpected, which yields a contradiction. If it was dropped due to Alternative b, then $C(d) \models_{\mathcal{T}} \exists R(d)$ and there is $\neg\exists R^-(c) \in \mathcal{N} \setminus \mathcal{A}'$. Then, $\exists x(R(d,x) \wedge (x \neq c))$ is unexpected, which yields a contradiction again.

Thus, we conclude with maximality of $\mathcal{A}_0$.

It remains to show the uniqueness of $\mathcal{A}_0$. Assume this is not that case and there is another maximal careful satisfiable subset $\mathcal{A}'_0$ of $\mathcal{A}$. Then, consider $C(a)$ that is in $\mathcal{A}'_0 \setminus \mathcal{A}_0$. Since both $\mathcal{A}'_0$ and $\mathcal{A}_0$ are satisfiable, they are subsets of $\mathcal{A}'$. Hence, $C(a) \in \mathcal{A}' \setminus \mathcal{A}_0$ and it was dropped from $\mathcal{A}'$ while constructing $\mathcal{A}_0$. Due to Lemma 4.8, there is an assertion $C'(b)$ in $\mathcal{N}$, such that $\{C(a), C'(b)\}$ is $\mathcal{T}$-entailing an unexpected role-constraining formula. Hence, $\mathcal{A}'_0 \setminus \mathcal{A}_0$ is empty. Therefore, $\mathcal{A}'_0 \subseteq \mathcal{A}_0$, which contradicts maximality of $\mathcal{A}'_0$. □

Finally, we can exhibit the algorithm CSAlg, which computes **CS**-evolutions. To this effect we have to define preclosure of ABoxes which will be used to detect unexpected RCFs.

Definition 4.6 (Preclosure)**.** The *preclosure of $\mathcal{A}$ wrt $\mathcal{T}$*, denoted $Precl_{\mathcal{T}}(\mathcal{A})$, is a subset of $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ obtained as follows: one removes from $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ all the assertions of the form $\exists R(a)$, whenever there is an assertion of the form $R(a,c)$ in $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$, for some constant c.

The algorithm CSAlg (see Fig. 7) takes as input an evolution setting and, first, computes the evolution wrt **BS**. Then, it computes the set UF of assertions that cause *unexpectedness* in $\mathsf{CSAlg}(\mathcal{E})$ (where $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ and $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$) and belong to $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$. Then it removes UF from $\mathsf{CSAlg}(\mathcal{E})$ by means of $\mathsf{Weeding}$ (see the definition of $\mathsf{Weeding}$ in Algorithm 4).

We conclude the section with correctness of CSAlg.

Theorem 4.10. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a DL-Lite$_{core}$-evolution setting, where $\mathcal{K} = \mathcal{T} \cup \mathcal{N}$. Then $\mathcal{T} \cup \mathsf{CSAlg}(\mathcal{E}) \cup \mathcal{N}$ is a **CS**-evolution for $\mathcal{E}$. Moreover, $\mathsf{CSAlg}(\mathcal{E})$ can be computed in time polynomial in $|\mathcal{E}|$.*

Proof. Follows from the the proof of Theorem 4.9, since it exactly mimics the construction of the proof. First, CSAlg computes the maximal subset $\mathcal{A}^c$ of $\mathcal{A}$ satisfiable with $\mathcal{N}$ (Line 1). Second, it detects those assertions that cause unexpectedness of Type (ii) but not of Type (i), according to Lemma 4.5. Following the proof of Theorem 4.9, CSAlg considers two cases to modify $\mathcal{A}^c$:

Algorithm 7: $\mathsf{CSAlg}(\mathcal{E})$

INPUT : *DL-Lite$_{core}$*-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$
OUTPUT : finite set of membership assertions $\mathcal{A}^c$

1 $\mathcal{A}^c := \mathsf{BSAlg}(\mathcal{E})$, $\mathsf{UF} := \emptyset$;
2 **for each** $\exists R(a) \in \mathit{Precl}_{\mathcal{T}}(\mathcal{N})$ **do**
3   **if** $R(a, b) \notin \mathcal{A}^c$ *for every* b **then**
4   **for each** $\exists R^- \sqsubseteq \neg C \in \mathsf{cl}(\mathcal{T})$ **do**
5             $\mathsf{UF} := \mathsf{UF} \cup \{C(d)\}$
6   **end**
7   **end**
8 **end**
9 **for each** $\exists R(a) \in \mathcal{A}^c \setminus \mathit{Precl}_{\mathcal{T}}(\mathcal{N})$ **do**
10   **if** $R(a, b) \notin \mathcal{A}^c$ *for every* b **then**
11   **if** *there is a concept* C *in* $\mathcal{T} \cup \mathcal{A} \cup \mathcal{N}$ *s.t.* $(\exists R^- \sqsubseteq \neg C) \in \mathsf{cl}(\mathcal{T})$ ***and*** $C(d) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \setminus \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ *for some* d **then**
12             $\mathsf{UF} := \mathsf{UF} \cup \{\exists R(a)\}$
13   **end**
14   **end**
15 **end**
16 $\mathcal{A}^c := \mathsf{Weeding}(\mathcal{T}, \mathcal{A}^c, \mathsf{UF})$;
17 **return** $\mathcal{A}^c$.

when $\exists R(a) \in \mathcal{N}$, and when $\exists R(a) \in \mathcal{A} \setminus \mathcal{N}$. In the former case (Lines 2-8) it checks that this unexpectedness is not of Type (i) (Line 3), and if it is not CSAlg deletes those assertions of $\mathcal{A}^c$ that lead to unexpected facts by adding them to the set UF (Line 4-5) of assertions to be deleted later (Line 16). In the latter case (Lines 9-15), CSAlg again checks the type of unexpectedness (Line 10), if it is of Type (i), it deletes $\exists R(a)$ (Lines 11-12) by means of $\mathsf{Weeding}$ (Line 16).

Polynomiality directly follows from the fact that building cl and *Precl* takes polynomial time, and running algorithm $\mathsf{Weeding}$ takes polynomial time as well. □

We illustrate **CS**-evolution on the following example.

Example 4.4. Continuing with $\mathcal{E}_0$ of Example 4.1, $\mathcal{K}_0 \diamond_{\mathbf{CS}} \mathcal{N}_0 = \mathcal{K}_0 \diamond_{\mathbf{BS}} \mathcal{N}_0 \setminus \{\exists \mathit{HasHusb}^-(\mathit{mary})\}$, that is,

$$\mathcal{K}_0 \diamond_{\mathbf{CS}} \mathcal{N}_0 = \mathcal{T}_0 \cup \{\mathit{Priest}(\mathit{john}), \mathit{Priest}(\mathit{pedro}), \mathit{Woman}(\mathit{mary})\}.$$

As expected, **CS**-evolution divorced Mary from John as soon as John becomes a priest. Note that the fact that the Mary is not married does not prevent her from being a woman, i.e., $\mathit{Woman}(\mathit{mary})$ which was entailed from $\mathcal{K}_0$ is still entailed from $\mathcal{K}_0 \diamond_{\mathbf{CS}} \mathcal{N}_0$, which is expected and intuitive. ■

4.3 Summary on Formula-Based Evolution

We now summarize on how one can use the results of this section to do ABox evolution of *DL-Lite$_{core}$* KBs under formula-based semantics in practice. There are basically two different semantics that one can use: **BS** and **CS**. Each of them has its positive and negative sides and one has to see which one is more appropriate for her target application. Both semantics can be computed in polynomial time: BSAlg can be used to compute **BS** evolution (see Theorem 4.2) and CSAlg to compute **CS** evolution (see Theorem 4.10).

5 Repairing Artifact-Centric Transition Systems

In this section, we discuss how we can apply repair-based semantic of KB evolution, such as those studied in Sections 3 and 4 to the setting where the knowledge to be evolved is the data component of a Knowledge and Action Base (KAB) of an Artifact Based System. We refer to Deliverable D2.4 [8] for a detailed treatment of KABs with full definitions of their syntax and semantics. Instead, we focus here on those aspects of KABs that are of relevance for applying an evolution based semantics, and consider therefore a simplified form of KAB, in which we assume that the process component is the universal process that allows every action to be executed at every step. For this reason, we entirely omit the process component in the specification of KABs, and formalize them as specified next.

5.1 Knowledge and Action Bases

A *Knowledge and Action Base (KAB)* is a triple $\mathcal{K} = (\mathcal{T}, S_0, \Gamma)$, where
(i) $\mathcal{T}$ is a TBox
(ii) S_0 is am ABox, i.e., a set of facts representing the initial information – the *initial state* of the system, and
(iii) Γ is a *set of actions* that specifies how the states of the system should evolve. An *action* $\gamma \in \Gamma$ is a set of effect specifications of the form $q_i(\bar{x}) \rightsquigarrow S_i(\bar{x})$, where $q_i(\bar{x})$ is a query with output variables $\bar{x}$ expressed over (the alphabet of) $\mathcal{T}$, and $S_i(\bar{x})$ is a set of atoms over (the alphabet of) $\mathcal{T}$ and $\bar{x}$. The actions might acquire external information by means of service calls, which are modeled through function symbols.

We illustrate KABs on the following example.

Example 5.1. Consider the KAB $\mathcal{K}_e = (\mathcal{T}_e, S_0, \Gamma_e)$ with $\Gamma_e = \{\gamma_1, \gamma_2\}$ and

$$\begin{aligned}
\mathcal{T}_e &= \{(\mathsf{funct}\ \mathsf{marriedTo}),\ \mathsf{De} \sqsubseteq \neg \mathsf{It}\},\\
S_0 &= \{\mathsf{Married}(\mathsf{Mariano})\},\\
\gamma_1 &= \mathtt{CA} \cup \{\mathsf{Married}(x) \rightsquigarrow \{\mathsf{marriedTo}(x, \mathbf{roG}(x)),\ \mathsf{De}(\mathbf{roG}(x))\}\},\\
\gamma_2 &= \mathtt{CA} \cup \{\mathsf{Married}(x) \rightsquigarrow \{\mathsf{marriedTo}(x, \mathbf{roI}(x)),\ \mathsf{It}(\mathbf{roI}(x))\}\}.
\end{aligned}$$

where CA (which stands for "copy all") is an operator that copies all the atoms of the current state to the new one. Intuitively, $\mathcal{T}_e$ says that a person cannot have more than one spouse (the relation marriedTo is functional) and that Germans are not Italians (and vice-versa); the initial state S_0 states that Mariano is married. Finally, Γ_e says that if a person, say x, is married (i.e., if $\mathsf{Married}(x)$ is satisfied) then one should explicitly add in the new state a fact about *(i)* the marriage of x , i.e., add $\mathsf{marriedTo}(x, \cdot)$ atom to the new state, where the name of his/her spouse can be found via a service call in a registry office in Germany (by calling a service $\mathbf{roG}(x)$ as in γ_1), or in Italy (by calling a service $\mathbf{roI}(x)$ as in γ_2); and *(ii)* the nationality of the x's wife, i.e., the wife is either German, according the action γ_1, or italian, according γ_2. ∎

Intuitively, an application of an action γ to a state S returns a new state S' defined as follows. Starting from $S' = \emptyset$, for each action $q_i(\bar{x}) \rightsquigarrow S_i(\bar{x})$ in γ, evaluate q_i over $\mathcal{T} \cup S$ (using the *certain answers* semantics [11]) and add all elements of $S'_i(\bar{a})$ to S' for every $\bar{a} \in cert(q_i, \mathcal{T} \cup S)$, where $cert(q_i, \mathcal{T} \cup S)$ denotes the certain answers of q_i over $\mathcal{T} \cup S$ and $S'_i(\bar{a})$ is the set of ABox assertions resulting from substituting $\bar{x}$ with $\bar{a}$ in $S'_i(\bar{x})$. We refer to [5, 7] for details and illustrate the definitions with an example.

Example 5.2. Continuing with Example 5.1, the following state S_1 can be obtained from S_0 by applying γ_1 as follows:

$$S_1 = \{\mathsf{Married}(\mathsf{Mariano}),\ \mathsf{marriedTo}(\mathsf{Mariano}, \mathbf{roG}(\mathsf{Mariano})), \mathsf{De}(\mathbf{roG}(\mathsf{Mariano}))\}.$$

Note that the query $\mathsf{Married}(x)$ of γ_1 evaluated over S_0 gives the only tuple Mariano. ∎

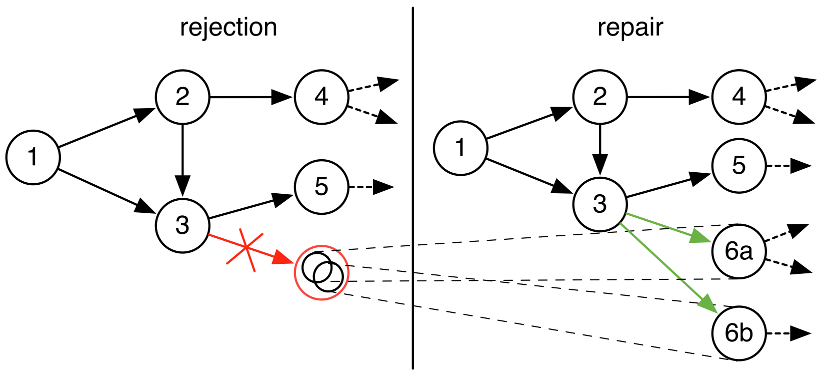

Figure 6 – A possible reject-inconsistency transition system for Example 5.1

In our study of KABs, we focus on TBoxes expressed in the *DL-Lite* family of description logics [11, 10]. We study *DL-Lite* languages which allow to express, in particular, functionality of direct and inverse object properties, and equality between constants. Note that, as in OWL 2 QL, and differently from traditional *DL-Lite*, we do not assume the unique name assumption to hold.

5.2 Transition System of a KAB

The transition system associated with a KAB $\mathcal{K} = (\mathcal{T}, S_0, \Gamma)$ is a graph $\mathcal{G}(\mathcal{K})$ whose nodes are states, reachable from S_0 by "executing" actions from Γ, and that are consistent with $\mathcal{T}$. More formally, (S, S') is an edge of $\mathcal{G}(\mathcal{K})$ if and only if *(i)* S' can be obtained from S by applying some action $\gamma \in \Gamma$, and *(ii)* $S' \cup \mathcal{T}$ is a consistent KB. In Figure 6, left, there is a (fragment of) $\mathcal{G}(\mathcal{K}_e)$ for $\mathcal{K}_e$ of Example 5.1. State 1 is initial and States 2-6 are reachable from it. Here we assume that States 2-5 are consistent with $\mathcal{T}_e$ and State 6 is not; thus, State 6 is "rejected" from $\mathcal{G}(\mathcal{K}_e)$ and there is no edge from State 3 to 6.

Example 5.3. Continuing with Example 5.2, the application of γ_2 to S_1 yields the state

$$S_2 = S_1 \cup \{\mathsf{marriedTo}(\mathsf{Mariano}, \mathbf{roI}(\mathsf{Mariano})),\ \mathsf{It}(\mathbf{roI}(\mathsf{Mariano}))\}.$$

We have that $S_2 \cup \mathcal{T}$ is inconsistent and hence S_2 is rejected from $\mathcal{G}(\mathcal{K}_e)$. The inconsistency comes from the fact that S_2 includes both $\mathsf{De}(\mathbf{roG}(\mathsf{Mariano}))$ and $\mathsf{It}(\mathbf{roI}(\mathsf{Mariano}))$, which together with $\mathsf{De} \sqsubseteq \neg\mathsf{It} \in \mathcal{T}_e$ leads to $\mathbf{roG}(\mathsf{Mariano}) \neq \mathbf{roI}(\mathsf{Mariano})$. At the same time, S_2 contains both $\mathsf{marriedTo}(\mathsf{Mariano}, \mathbf{roG}(\mathsf{Mariano}))$ and $\mathsf{marriedTo}(\mathsf{Mariano}, \mathbf{roI}(\mathsf{Mariano}))$, and due to functionality of $\mathsf{marriedTo}$ in $\mathcal{T}_e$ this yields that $\mathbf{roG}(\mathsf{Mariano}) = \mathbf{roI}(\mathsf{Mariano})$. ∎

In [7, 5] it was shown that even checking (simple) propositional LTL safety properties over transition systems represented by $\mathcal{G}(\mathcal{K})$ is in general undecidable. It was also shown that a specific form of weak-acyclicity condition on the action specification of KABs (inspired by weak-acyclicity in data-exchange [24]), is sufficient to guarantee decidability. We argue here, however, that the way in which $\mathcal{G}(\mathcal{K})$ is defined in [7, 5] is too restrictive, since for many applications it is desirable not to immediately reject states that are inconsistent with $\mathcal{T}$. Indeed, the inconsistency may be due to a possibly very small portion of the ABox, so in this case one might want to allow for the action generating the inconsistent state to be executed, while "repairing" the generated inconsistency. Consider also that inconsistencies may arise from the information brought into the state by calls to external services. These are out of the control of the system, so that conflicts of knowledge may be unavoidable when the external information is *integrated* with the one coming from the state in which the action was applied.

5.3 Repairing Inconsistent States

In Example 5.3, the state S_2 is rejected because some person would have to be both Italian and German, which contradicts the TBox. However, the inconsistency is caused only by a (small) portion of the ABox, and therefore it would be desirable not to lose the remaining consistent part and keep this state, *repairing* it beforehand. Indeed, it could be the case that the wife of Mariano used to be an Italian, but then she moved to Germany and changed her citizenship (or the other way around), while the information in the registry offices has not been updated. We do not have control over the offices, but we can repair the state correspondingly to the options we have, trying to remove the inconsistency while keeping at the same time as much information as possible.

Example 5.4. The following states are possible repairs of S_2 from Example 5.3:

$$\begin{aligned} S^1_{rep} = \{&\mathbf{roG}(\mathsf{Mariano}) = \mathbf{roI}(\mathsf{Mariano}),\ \mathsf{Married}(\mathsf{Mariano}),\\ &\mathsf{De}(\mathbf{roG}(\mathsf{Mariano})),\ \mathsf{marriedTo}(\mathsf{Mariano}, \mathbf{roG}(\mathsf{Mariano}))\},\\ S^2_{rep} = \{&\mathbf{roG}(\mathsf{Mariano}) = \mathbf{roI}(\mathsf{Mariano}),\ \mathsf{Married}(\mathsf{Mariano}),\\ &\mathsf{It}(\mathbf{roI}(\mathsf{Mariano})),\ \mathsf{marriedTo}(\mathsf{Mariano}, \mathbf{roI}(\mathsf{Mariano}))\}. \end{aligned}$$

The repair S^1_{rep} represents the case when Mariano's wife is German, and S^2_{rep} – Italian. ■

In Figure 6, right, we present a *repair-based* execution flow, where instead of rejecting the inconsistent State 6, we set as its successors its repairs, namely State 6a and State 6b, and continue to execute Γ.

6 Conclusion

We studied model-based approaches to ABox evolution (update and revision) over $\textit{DL-Lite}_{core}$ and its fragment $\textit{DL-Lite}^{pr}$, which extends (the first-order fragment of) RDFS. $\textit{DL-Lite}^{pr}$ is closed under most of the MBAs, while $\textit{DL-Lite}_{core}$ is *not* closed under any of them. We showed that if the TBox of $\mathcal{K}$ entails a pair of assertions of the form $\mathcal{A} \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq \neg C$, then an interplay of $\mathcal{N}$ and $\mathcal{A}$ may lead to inexpressibility of $\mathcal{K} \diamond_S \mathcal{N}$ with $\mathcal{S} = \mathbf{L}^a_{\subseteq}$.

For $\textit{DL-Lite}^{pr}$ we provided the algorithms to compute evolution results for six model-based approaches and approximate evolution for the remaining two. For $\textit{DL-Lite}_{core}$ (under some restrictions) we studied the properties of evolution under a local model-based approach $\mathbf{L}^a_{\subseteq}$. In particular, we introduced the notion of prototypal sets that extends the notion of canonical models. We proved that prototypal sets for $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$ exist, and that they are of exponential size in $|\mathcal{K} \cup \mathcal{N}|$, and showed an abstract procedure that constructs them. Based on the insights gained, we proposed a polynomial time algorithm to compute a minimal sound $\textit{DL-Lite}_{core}$-approximation of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$. We also believe that prototypes are important since they can be used to study evolution for ontology languages other than $\textit{DL-Lite}_{core}$. In general, we provided some understanding on why *DL-Lite* is not closed under MBAs to evolution, and what are the properties of sets of models $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$. This understanding is a prerequisite to proceed with the study of evolution in more expressive DLs and to understand what to expect from MBAs in such logics.

We also studied formula-based approaches to ABox evolution. We reviewed known classical formula-based semantics and introduced Bold Semantics. We proved uniqueness of ABox evolution under the latter semantics and presented an efficient algorithm to compute it. Finally, we introduced Careful Semantics, proved uniqueness of ABox evolution under Careful Semantics and present an efficient algorithm to compute it.

6.1 Further Work in ACSI Task 2.5

We believe that we have gained a thorough understanding of both formula and model-based semantics of ABox evolution. The next step will be to apply these insights to Knowledge and Action Bases as specified in Section 5 and to Semantically-governed Artifact Systems (cf. Deliverable D2.3 [13]). More precisely, we are currently working on various aspects related to the specification of the semantics of KABs considering a repair-based semantics for the KB component, and the verification of temporal properties in this setting. In particular, we are interested in investigating how to extend to this new setting the decidability results established for verification of μ-calculus properties over Data Centric Dynamic Systems [6, 7, 8] and KABs [5, 8]. Moreover, we intend to pursue another relevant and challenging direction, namely studying knowledge evolution at the TBox level, again with a repair-based semantics. This type of evolution is of particular relevance to address changes in the interoperation hub at the level of the specification, and not only at the level of the data.

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A Appendix: Proofs

A.1 Proofs of Section 3.2.1

In this and the following section we will need the following property of *DL-Lite*$_{core}$.

Proposition A.1. *Let $\mathcal{T} \cup \mathcal{A}$ be a satisfiable DL-Lite$_{core}$ KB and L be a membership assertion. If $\mathcal{A} \models_{\mathcal{T}} L$, then there exists a membership assertion $L_0 \in \mathcal{A}$ such that $L_0 \models_{\mathcal{T}} L$.*

Proof. Assume that L is a positive assertion, i.e., of the form $P(a,b)$, $A(c)$, or $\exists R(c)$. Since $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ is a model of $\mathcal{T} \cup \mathcal{A}$ [11], the entailment $\mathcal{A} \models_{\mathcal{T}} L$ implies that $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ models L. Suppose that $L \in \mathcal{A}$, then, by taking $L_0 = L$, the lemma trivially holds. Suppose that $L \notin \mathcal{A}$, then we have that either $L \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ with $a, b, c \in \mathsf{adom}(\mathcal{K})$, when $L \in \{P(a,b), A(c)\}$, or $R(c,x) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ with $c \in \mathsf{adom}(\mathcal{K})$ and $x \notin \mathsf{adom}(\mathcal{A})$, when $L = \exists R(c)$. By the definition of chase, for every atom in $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, there is a sequence of atoms $f_1, \dots, f_n$, where *(i)* $f_n = L$ or $f_n = R(c,x)$, depending on the shape of L; *(ii)* $f_1 \in \mathcal{A}$, or $\exists R'(c') \in \mathcal{A}$ and $f_1 = R'(c', x')$, where $x' \notin \mathsf{adom}(\mathcal{A})$; *(iii)* each f_{i+1} is derivable from f_i by triggering a positive inclusion assertion of $\mathcal{T}$, that is, $f_i \models_{\mathcal{T}} f_{i+1}$. Due to transitivity of $\models_{\mathcal{T}}$, due to $f_1 \models_{\mathcal{T}} f_n$, and by taking $L_0 = f_1$ or $L_0 = \exists R'(c')$ depending on the shape of f_1, we obtain $L_0 \models_{\mathcal{T}} L$ and conclude the proof.

Assume that L is a negative inclusion assertion of the form $\neg A(c)$. If $L \in \mathcal{A}$, then, by taking $L_0 = L$, we conclude the proof. Assume that $L \notin \mathcal{A}$. Assume that

$$\text{for every assertion } L' \in \mathcal{A} \text{ it holds } L' \not\models_{\mathcal{T}} L. \tag{27}$$

Let $L_1, \dots, L_n$ be all the PIs of $\mathcal{A}$. Consider the interpretation:

$$\mathcal{I} = \bigcup_{i=1}^{n} \mathsf{chase}_{\mathcal{T}}(L_i) \cup \mathsf{chase}_{\mathcal{T}}(A(c)).$$

Clearly, $\mathcal{I} \not\models L$. We now show that $\mathcal{I} \models \mathcal{A} \cup \mathcal{T}$, so we will obtain a contradiction with $\mathcal{A} \models_{\mathcal{T}} L$. Observe that $\mathcal{I}$ is a model of $\mathcal{A}$. Indeed, it models all the positive MAs of $\mathcal{A}$ by construction. Each $\mathsf{chase}_{\mathcal{T}}(L_i)$ (and consequently their union) satisfies all negative MAs of $\mathcal{A}$. Assume there is i for which $\mathsf{chase}_{\mathcal{T}}(L_i)$ does not satisfy an negative MA $\neg g$ of $\mathcal{A}$. Thus, $\{L_i, \neg g\} \models_{\mathcal{T}} \bot$ which contradicts satisfiability of $\mathcal{T} \cup \mathcal{A}$. Finally, $\mathsf{chase}_{\mathcal{T}}(A(c))$ satisfies all negative MAs of $\mathcal{A}$. Assume it is not the case, and there is a negative MA $\neg g \in \mathcal{A}$ such that $\mathsf{chase}_{\mathcal{T}}(A(c)) \models g$. Then, $A(c) \models_{\mathcal{T}} g$, and $\neg g \models_{\mathcal{T}} \neg A(c)$, thus we found an assertion in $\mathcal{A}$ that $\mathcal{T}$-entails L, which contradicts Equation 27. Clearly, $\mathcal{I}$ models all the PIs of $\mathcal{T}$. It remains to show that $\mathcal{I}$ models each NI of $\mathcal{T}$. Assume there is a NI α such that $\mathcal{I} \not\models \alpha$. Then, there are two atoms f and f' in $\mathcal{I}$ such that $f \to \neg f'$ is an instantiation of the first-order interpretation of α. This implies that $\{f, \alpha\} \models \neg f'$. Clearly, ABoxes $\{L_i\}$ for $1 \leq i \leq n$ and $\{A(c)\}$ satisfy α, so does $\mathsf{chase}_{\mathcal{T}}(L_i)$ for $1 \leq i \leq n$ and $\mathsf{chase}_{\mathcal{T}}(A(c))$ due to Lemma 12 of [11]. This implies that $\{f, f'\} \not\subseteq \mathsf{chase}_{\mathcal{T}}(L_i)$ for each $1 \leq i \leq n$ and $\{f, f'\} \not\subseteq \mathsf{chase}_{\mathcal{T}}(A(c))$. Thus, two cases are possible:

(i) $f \in \mathsf{chase}_{\mathcal{T}}(L_i)$ for some $i \in \{1, \dots, n\}$ and $f' \in \mathsf{chase}_{\mathcal{T}}(A(c))$ (the case when $f' \in \mathsf{chase}_{\mathcal{T}}(L)$ and $f \in \mathsf{chase}_{\mathcal{T}}(A(c))$ is symmetric),

(ii) $f \in \mathsf{chase}_{\mathcal{T}}(L_i)$ and $f' \in \mathsf{chase}_{\mathcal{T}}(L_j)$ for some different $i, j \in \{1, \dots, n\}$.

In Case (i), $f' \in \mathsf{chase}_{\mathcal{T}}(A(c))$ implies that $\neg f' \models_{\mathcal{T}} \neg A(c)$. Combining the latter entailment with $\{f, \alpha\} \models \neg f'$ we obtain $f \models_{\mathcal{T}} \neg A(c)$. Since $L_i \models_{\mathcal{T}} f$, we conclude that $L_i \models_{\mathcal{T}} \neg A(c)$ which contradicts the assumption in Equation 27 and concludes the proof. In Case (ii), analogously to Case (i), we conclude that $L_i \models_{\mathcal{T}} \neg L_j$, thus A does not satisfy α which yields a contradiction with satisfiability of $\mathcal{T} \cup \mathcal{A}$.

The case when $L = \neg\exists R(c)$ is analogous to the previous one. □

Proof of Proposition 3.1.

It follows from the definition of AtAlg and the facts that in *DL-Lite*^{pr} disjointness that involve roles or their projections is forbidden and $\mathcal{N}$ contains only positive membership assertions. Indeed, let $\mathcal{A} \models_{\mathcal{T}} R(a,b)$ and $\mathsf{AtAlg}(\mathcal{K},\mathcal{N}) \cup \mathcal{N} \not\models_{\mathcal{T}} R(a,b)$. Then, $\{R(a,b)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$ (see Algorithm 1). Thus, there are membership assertions L_1 and L_2, and a NI $\alpha \in \mathsf{cl}(\mathcal{T})$ such that $\{R(a,b)\} \cup \mathcal{N} \models_{\mathcal{T}} \{L_1, L_2\}$ and $\alpha \models L_1 \to \neg L_2$. Note that $L_1 \to \neg L_2$ should be seen as a first order formula with two subformulas L_1 and L_2, both without free variables. The semantics of this formula is defined straightforwardly: $\mathcal{I} \models L_1 \to \neg L_2$ if $\mathcal{I} \models L_1$ and $\mathcal{I} \not\models L_2$ for every interpretation $\mathcal{I}$. Due to Proposition A.1, one of the following two cases holds: $R(a,b) \models_{\mathcal{T}} L_1$ or $R(a,b) \models_{\mathcal{T}} L_2$. Consider the first case (the second one is symmetric). Combining $R(a,b) \models_{\mathcal{T}} L_1$ and $\alpha \models L_1 \to \neg L_2$ we obtain that $\alpha \models R(a,b) \to \neg L_2$. Thus, α is of the form $\exists R \sqsubseteq \neg B$ or $\exists R^- \sqsubseteq \neg B$ for some basic concept B. Either case contradicts the fact that $(\mathcal{T} \cup \mathcal{A})$ is a *DL-Lite*^{pr} KB. Similarly, the case when $\mathcal{A} \models_{\mathcal{T}} \exists R(a)$ can be proved. □

Proof of Proposition 3.2.

$\Leftarrow$. Assume that for every $f_1 \in \mathcal{I}_1$ and $f_2 \in \mathcal{I}_2$ it holds that $\{f_1, f_2\} \not\models_{\mathcal{T}} \bot$, but $\mathcal{I}_1 \cup \mathcal{I}_2 \not\models \mathcal{T}$. Then, there is an assertion $\alpha \in \mathsf{cl}(\mathcal{T})$ such that $\mathcal{I}_1 \cup \mathcal{I}_2$ does not model α. Suppose α is a PI, then there is a ground atom g_1 in $\mathcal{I}_1 \cup \mathcal{I}_2$ satisfying the property: for every ground atom g_2 such that $g_1 \to g_2$ is an instantiation of the first-order translation of α, it holds that $g_2 \notin \mathcal{I}_1 \cup \mathcal{I}_2$. The fact that $g_1 \in \mathcal{I}_1 \cup \mathcal{I}_2$ implies that one of the two cases holds: *(i)* $g_1 \in \mathcal{I}_1$ or *(ii)* $g_1 \in \mathcal{I}_2$. From Case (i) together with $g_2 \notin \mathcal{I}_1 \cup \mathcal{I}_2$, we conclude that $\mathcal{I}_1 \not\models \alpha$, and from Case (ii) together with $g_2 \notin \mathcal{I}_1 \cup \mathcal{I}_2$, we conclude that $\mathcal{I}_2 \not\models \alpha$. Either case contradicts the fact that $\mathcal{I}_1$ and $\mathcal{I}_2$ are models of $\mathcal{T}$.

Suppose α is a NI, then, due to Lemma 12 (more precisely, its straightforward extended to the case when $\mathcal{A}$ a possibly infinite set of atoms) of [16], there are two atoms f_1 and f_2 in $\mathcal{I}_1 \cup \mathcal{I}_2$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. Since $\mathcal{I}_1 \models \alpha$ and $\mathcal{I}_2 \models \alpha$, neither $\{f_1, f_2\} \subseteq \mathcal{I}_1$ nor $\{f_1, f_2\} \subseteq \mathcal{I}_2$ holds. Thus, either of the two cases holds: $f_1 \in \mathcal{I}_1$ and $f_2 \in \mathcal{I}_2$, or $f_2 \in \mathcal{I}_1$ and $f_1 \in \mathcal{I}_2$. Either case contradicts the assumption of the "if" direction.

$\Rightarrow$. Assume that there are $f_1 \in \mathcal{I}_1$ and $f_2 \in \mathcal{I}_2$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. Then, $\mathcal{I}_1 \cup \mathcal{I}_2 \not\models \mathcal{T}$, which contradicts the assumption of the "only if" direction. □

Proof of Proposition 3.3.

Assume that there is a general MA g such that $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \not\models \mathcal{T}$. Then, there is an assertion $\alpha \in \mathsf{cl}(\mathcal{T})$ s.t.

$$\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \not\models \alpha. \tag{28}$$

Assume that α is an NI. Clearly, if a set of atoms satisfies a negative inclusion assertion, then any subset of this set of atoms does so. This implies that, since $\mathcal{I} \models \alpha$ and $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \subseteq \mathcal{I}$, $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \models \alpha$, which contradicts the assumption in Equation 28.

Assume that g is a positive MA and α is a PI. Then, Equation 28 implies that there is a ground atom f_1 in $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ satisfying the property: for every ground atom f_2 such that $f_1 \to f_2$ is an instantiation of the first-order translation of α, $f_2 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$. Observe that $f_1 \in \mathcal{I}$ and $\mathcal{I} \models \alpha$, thus at least one such f_2, say $\hat{f}_2$, is in $\mathcal{I}$. Since $\hat{f}_2 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$, we have that $\hat{f}_2 \in \mathsf{root}_{\mathcal{T}}(g)$. Therefore, by the definition of $\mathsf{root}_{\mathcal{T}}(g)$, $f_1 \in \mathsf{root}_{\mathcal{T}}(g)$ and $f_1 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$, which contradicts the fact that $f_1 \in \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ and concludes the proof.

Assume that g is a negative MA and α is a PI. Let α be $B \sqsubseteq B'$. By exactly the same reason as the case of positive g, there are the atoms f_1 and $\hat{f}_2$ such that $f_1 \to \hat{f}_2$ instantiate $B \sqsubseteq B'$. Since $\hat{f}_2 \in \mathsf{root}_{\mathcal{T}}(g)$, there is an NI of the form $B' \sqsubseteq \neg B''$ such that $\mathcal{T} \models B' \sqsubseteq \neg B''$ and $\hat{f}_2 \to g$ is its instantiation. From $\mathcal{T} \models B' \sqsubseteq \neg B''$ and $\mathcal{T} \models B \sqsubseteq B'$, we conclude that $\mathcal{T} \models B \sqsubseteq \neg B''$ and therefore $f_1 \in \mathsf{root}_{\mathcal{T}}(g)$ which contradicts the fact that $f_1 \in \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ and concludes the proof. □

Proposition A.2. *Let $(\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL-Lite^{pr}-evolution setting. Let $\mathcal{J}_0 \in \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^a_{\#}$, $\mathcal{I}_0 \in \mathsf{Mod}(\mathcal{K})$, and $\mathcal{I}_0$ is $\mathbf{G}^a_{\#}$-minimally distant from $\mathcal{J}_0$. Then, $|\mathcal{I}_0 \ominus \mathcal{J}_0|$ is finite.*

Proof. Let $S = \mathbf{G}^a_{\#}$. Suppose that $|\mathcal{I}_0 \ominus \mathcal{J}_0|$ is infinite. If there exist models $\mathcal{I}' \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J}' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $|\mathcal{I}' \ominus \mathcal{J}'|$ is finite, this will contradict the fact that $\mathcal{J}_0 \in \mathcal{K} \diamond_S \mathcal{N}$ since $|\mathcal{I}' \ominus \mathcal{J}'| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. It is easy to see that these $\mathcal{I}'$ and $\mathcal{J}'$ always exist. Indeed, since *DL-Lite^{pr}* is a sub-language of *DL-Lite$_{core}$* and *DL-Lite$_{core}$* enjoys the final model property, one can choose finite models $\mathcal{I}'$ and $\mathcal{J}'$ (for them $\mathcal{I}' \ominus \mathcal{J}'$ is clearly a finite set). Then, $|\mathcal{I}' \ominus \mathcal{J}'| \leq |\mathcal{I}' \cup \mathcal{J}'|$, i.e., $|\mathcal{I}' \ominus \mathcal{J}'|$ is finite. □

A.2 Proofs of Section 3.2.2

Proof of Proposition 3.6.
Due to Proposition 3.2, it suffices to show that for every $f_1 \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))$ and $f_2 \in \mathcal{J}[A(c)]$ we have $\{f_1, f_2\} \not\models_{\mathcal{T}} \bot$. Assume this is not the case, that is, there is $f_1 \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))$ and $f_2 \in \mathcal{J}[A(c)]$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$.

We now show that

$$\{f_1, A(c)\} \models_{\mathcal{T}} \bot. \tag{29}$$

From $\{f_1, f_2\} \models_{\mathcal{T}} \bot$ it clearly follows that there is an NI of the form $A_1 \sqsubseteq \neg A_2$ such that $\mathcal{T} \models A_1 \sqsubseteq \neg A_2$, and there are atoms $A_1(d) \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))[f_1]$ and $A_2(d) \in \mathcal{J}[A(c)]$. We do not need to consider NIs that involve roles since we are in the case of *DL-Lite^{pr}*. From $A_2(d) \in \mathcal{J}[A(c)]$ we conclude that $d = c$. Indeed, from $A_2(d) \in \mathcal{J}[A(c)]$ we conclude that there is a sequence of atoms $f_1, \dots, f_n$, where *(i)* $f_1 = A(c)$; *(ii)* $f_n = A_2(d)$, *(iii)* each f_{i+1} is derivable from f_i by triggering a positive inclusion assertion α_i of $\mathsf{cl}(\mathcal{T})$. If $c \neq d$, then there is a role symbol occurring in at least one α_i. Indeed, if each α_i has no role symbol, then due to transitivity of $\models_{\mathcal{T}}$, we have $f_1 \models_{\mathcal{T}} f_n$ and therefore $c = d$. Let R be a role symbol occurring in α_j with the highest index, that is, $j = n$, or $j < n$ and for each α_i where $j < i < n$ there is no role symbol occurring in α_i. Then, α_j is of the form $\exists R \sqsubseteq A'$. Thus, $\mathcal{T} \models_{\mathcal{T}} \exists R \sqsubseteq A_2$. Combining this with $\mathcal{T} \models A_1 \sqsubseteq \neg A_2$, we obtain that $\mathcal{T} \models_{\mathcal{T}} \exists R \sqsubseteq \neg A_1$, which contradicts the fact that $\mathcal{T}$ is in *DL-Lite^{pr}*. Thus, there are no role symbols in each α_t, where $1 \leq t \leq n$. Therefore, $c = d$ and $A(c) \models_{\mathcal{T}} A_2(c)$.

Analogously, one can show that $A_1(c) \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))[f_1]$ implies that $f_1 \models_{\mathcal{T}} A_1(c)$. To sum up, we proved that

$$f_1 \models_{\mathcal{T}} A_1(c), \quad A(c) \models_{\mathcal{T}} A_2(c), \quad A_1(c) \models_{\{A_1 \sqsubseteq \neg A_2\}} \neg A_2(c),$$

thus Equation 29 holds.

Since for every $f_2 \in \mathcal{J}[A(c)]$ it holds that $A(c) \models_{\mathcal{T}} f_2$, we have $\{f_1, A(c)\} \models_{\mathcal{T}} \bot$. Thus, $\{f_1, A(c)\} \models_{\mathcal{T}} \neg A(c)$. There are only two literals in $\{f_1, A(c)\}$ and $A(c) \not\models_{\mathcal{T}} \neg A(c)$. Thus, due to Proposition A.1, we conclude that $f_1 \models_{\mathcal{T}} \neg A(c)$. This contradicts the assumption that $f_1 \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))$ and concludes the proof. □

A.3 Proofs of Section 3.3

Proof of Theorem 3.10.
It remains to show the case $\mathbf{G}^s_{\#} \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_{\subseteq}$. Consider $\mathcal{M}_{\#} = \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{G}^s_{\#}$, which is based on the distance $\mathsf{dist}^s_{\#}$, and $\mathcal{M}_{\subseteq} = \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{G}^s_{\subseteq}$, which is based on $\mathsf{dist}^s_{\subseteq}$, for an evolution setting $(\mathcal{K}, \mathcal{N})$. We now are interested in establishing whether $\mathcal{M}_{\#} \subseteq \mathcal{M}_{\subseteq}$ holds. Assume $\mathcal{J}' \in \mathcal{M}_{\#}$ and $\mathcal{J}' \notin \mathcal{M}_{\subseteq}$. From the former assumption, we conclude existence of a model $\mathcal{I}'$ such that for every pair of models $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, it holds that $\mathsf{dist}^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0) \leq \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})$. From the latter assumption, $\mathcal{J}' \notin \mathcal{M}_{\subseteq}$, we conclude existence of models $\mathcal{I}'' \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J}'' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $\mathsf{dist}^s_{\subseteq}(\mathcal{I}'', \mathcal{J}'') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}', \mathcal{J}')$. Since the signature of $\mathcal{K} \cup \mathcal{N}$ is finite, the distance $\mathsf{dist}^s_{\subseteq}$ between every two models over this signature is

also finite. Thus, we obtain that $\mathsf{dist}^s_{\#}(\mathcal{I}'', \mathcal{J}'') \subsetneq \mathsf{dist}^s_{\#}(\mathcal{I}', \mathcal{J}')$, which contradicts the fact that $\mathcal{J}' \in \mathcal{M}_{\#}$ and concludes the proof. $\square$

Proof of Proposition 3.11.
Let $g = R(a,b)$, the case when $g = \exists R(a)$ is analogous. Assume $\mathcal{A} \models_{\mathcal{T}} R(a,b)$, while there is $\mathcal{J}_0 \in \mathcal{S}$ such that $\mathcal{J}_0 \not\models R(a,b)$. Let $\mathcal{I}_0$ be a model of $\mathcal{T} \cup \mathcal{A}$ such that $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$ under $\mathbf{L}^a_{\#}$. We now exhibit $\mathcal{J}'_0 \models \mathcal{T} \cup \mathcal{N}$ such that $|\mathcal{I}_0 \ominus \mathcal{J}'_0| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. Consider $\mathcal{J}'_0 = \mathcal{J}_0 \cup \mathcal{I}_0^{R(a,b)}$. Note that $\mathcal{A} \models_{\mathcal{T}} R(a,b)$ and therefore the set $\mathcal{I}_0^{R(a,b)}$ is not empty.

Observe that $\mathcal{J}'_0 \models \mathcal{T} \cup \mathcal{N}$. Indeed, $\mathcal{J}'_0 \models \mathcal{N}$ since $\mathcal{N}$ contains positive MAs only and $\mathcal{J}_0 \models \mathcal{N}$. $\mathcal{J}'_0$ models all PIs from $\mathcal{T}$ since both $\mathcal{J}_0$ and $\mathcal{I}_0^{R(a,b)}$ does so. Assume there is an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ of the form $A_1 \sqsubseteq \neg A_2$, where A_1 and A_2 are atomic, such that $\mathcal{J}'_0 \not\models \alpha$.[11] Thus, there is a pair of atoms $\{A_1(c), A_2(c)\} \subseteq \mathcal{J}'_0$. Observe that $\{A_1(c), A_2(c)\} \not\subseteq \mathcal{I}_0^{R(a,b)}$ and $\{A_1(c), A_2(c)\} \not\subseteq \mathcal{J}_0$. Indeed, the ABox $\{R(a,b)\}$ obviously satisfies α and due to Lemma 12 of [11] so does the model $\mathcal{I}_0^{R(a,b)}$, and therefore $\{A_1(c), A_2(c)\} \not\subseteq \mathcal{I}_0^{R(a,b)}$. Since $\mathcal{J}_0 \in \mathcal{S}$, it clearly holds that $\{A_1(c), A_2(c)\} \not\subseteq \mathcal{J}_0$. Therefore, one of the two cases holds: $A_1(c) \in \mathcal{J}_0$ and $A_2(c) \in \mathcal{I}_0^{R(a,b)}$, or $A_2(c) \in \mathcal{J}_0$ and $A_1(c) \in \mathcal{I}_0^{R(a,b)}$. Either case is possible since neither $\mathcal{J}_0$ nor $\mathcal{I}_0^{R(a,b)}$ is empty. Consider the first case, the second case is symmetric. The membership $A_2(c) \in \mathcal{I}_0^{R(a,b)}$ implies an existence of a sequence of atoms $f_1, \ldots, f_n$ in $\mathcal{I}_0^{R(a,b)}$ such that $n \geq 2$, $f_1 = R(a,b)$, $f_n = A_2(c)$, and for each $1 \leq i \leq (n-1)$ there is a PI $\alpha_1 \in \mathsf{cl}(\mathcal{T})$ such that $f_i \to f_{i+1}$ is an instantiation of α_i. We now show by induction on n that $\mathsf{cl}(\mathcal{T})$ contains an NI of the form $\exists R' \sqsubseteq \neg A'$ for some role R' and atomic concept A', which will give a contradiction with the fact that $\mathcal{T} \cup \mathcal{A}$ is a *DL-Lite*^{pr} KB. If $n = 2$ then $\alpha_1 = \exists R \sqsubseteq A_2$ or $\alpha_1 = \exists R^- \sqsubseteq A_2$. The former case combined with α gives that $\exists R \sqsubseteq \neg A_1 \in \mathsf{cl}(\mathcal{T})$ and the latter one: $\exists R^- \sqsubseteq \neg A_1 \in \mathsf{cl}(\mathcal{T})$. Thus, we obtain a contradiction. If $n > 2$, then consider α_{n-1}. The shape of α_{n-1} is either $A' \sqsubseteq A_2$ or $\exists R' \sqsubseteq A_2$. Combining the former case with α, we obtain that $A_1 \sqsubseteq \neg A'$ and we conclude the proof by the induction assumption. Combining the later case with α we obtain that $\exists R' \sqsubseteq \neg A_1$, which gives a contradiction. We conclude that $\mathcal{J}'_0 \models \mathcal{T} \cup \mathcal{N}$.

It remains to show that $|\mathcal{I}_0 \ominus \mathcal{J}'_0| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. By contraction, $\mathcal{I}_0 \ominus \mathcal{J}'_0 \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$. Since we are in *DL-Lite*^{pr}, $\mathcal{I}_0 \ominus \mathcal{J}'_0$ is finite. Thus, $|\mathcal{I}_0 \ominus \mathcal{J}'_0| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. $\square$

A.4 Proofs of Section 3.4.1

Proof of Proposition 3.17.
Recall that

$$\mathsf{func}(R,c) \doteq \forall x \forall y.\, (R(x,c) \land R(x,y) \to y = c)\,.$$

Assume there is a *DL-Lite*$_{\mathcal{R}}$ KB $\mathcal{K} = (\mathcal{T} \cup \mathcal{A})$ such that $\mathcal{K} \not\models \neg\exists R^-(c)$ and $\mathcal{K} \models \mathsf{func}(R,c)$. Let a and b be constants *not* occurring in $\mathcal{K}$. Consider $\mathcal{A}' = \mathcal{A} \cup \{R(a,c), R(a,b)\}$. We now show that $\mathcal{A}'$ satisfies all the NIs of $\mathsf{cl}(\mathcal{T})$. If this is the case, then due to Lemma 12 of [11], this observation gives that $\mathcal{T} \cup \mathcal{A}'$ is satisfiable. Let $\mathcal{I}$ be a model of $\mathcal{T} \cup \mathcal{A}'$. Since $\mathcal{I} \models \mathcal{A}'$, it holds that $\mathcal{I} \models \mathcal{A}$ and therefore $\mathcal{I} \models \{R(a,c), R(a,b)\}$, that is, $\mathcal{I}$ does not satisfy $\mathsf{func}(R,c)$. On the other hand, $\mathcal{I} \models \mathcal{K}$. This contradicts the fact that $\mathcal{K} \models \mathsf{func}(R,c)$.

So, it remains to show that $\mathcal{A}'$ satisfies all the NIs of $\mathsf{cl}(\mathcal{T})$. Assume there is an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ such that $\mathcal{A}'$ does not satisfy α. Then, there are two MAs f and g in $\mathcal{A}'$ such that $f \to \neg g$ is an instantiation of the first-order interpretation of α. Four cases are possible: *(i)* $\{f,g\} \subseteq \{R(a,c), R(a,b)\}$, *(ii)* $f \in \mathcal{A}$ and $g \in \{R(a,c), R(a,b)\}$, *(iii)* $g \in \mathcal{A}$ and $f \in \{R(a,c), R(a,b)\}$, and *(iv)* $\{f,g\} \in \mathcal{A}$. Assume that Case (i) holds. One possibility of $\{f,g\} \in \{R(a,c), R(a,b)\}$ is when $f = R(a,c)$ and $g = R(a,b)$. Then, $\alpha = \exists R \sqsubseteq \neg\exists R$, which contradicts the coherency of $\mathcal{K}$. Other possibilities of $\{f,g\} \in \{R(a,c), R(a,b)\}$ are analogous. Assume that Case (ii) holds.

[11]Note that $\mathcal{T} \cup \mathcal{A}$ is a *DL-Lite*^{pr} KB and therefore all NIs in $\mathsf{cl}(\mathcal{T})$ has only atomics concepts on the left and the right of $\sqsubseteq$.

Let $g = R(a, b)$. Since $f \to \neg g$ instantiates α, f and g should share at least one constant. Since $f \in \mathcal{A}$ and neither a nor b occurs in $\mathcal{A}$, this constant is c. Hence, $g = R(a, c)$. Therefore, α is of the form $B \sqsubseteq \neg R^-$, and $B(c) \in \mathcal{A}$. Thus, $\mathcal{K} \models \neg\exists R^-(c)$, which contradicts the assumptions of the proposition on $\mathcal{K}$. Case (iii) is analogous to Case (ii). Assume that Case (iv) holds. Thus, $\mathcal{A} \not\models \alpha$ and due to Lemma 12 of [11] $\mathcal{T} \cup \mathcal{A}$ is unsatisfiable, which contradicts the coherence of $\mathcal{K}$. □

A.5 Proofs of Section 3.4.2

In order to prove Lemma 3.20, we will show the following two technical propositions.

Proposition A.3. *Let $n \geq 2$ be a natural number, $\alpha_1 = B_1 \sqsubseteq B'_1, \ldots, \alpha_n = B_n \sqsubseteq B'_n$ DL-Lite$_\mathcal{R}$ PIs, $\beta = B \sqsubseteq \neg B'$ a DL-Lite$_\mathcal{R}$ NI, and $g_0, \ldots, g_n, f$ a sequence of ground atoms such that $g_{i-1} \to g_i$ for $1 \leq i \leq n$ instantiates α_i, and $g_n \to \neg f$ instantiates β. If $B_1 \sqsubseteq \neg B' \notin \mathsf{cl}(\{\alpha_1, \cdots, \alpha_n, \beta\})$, then at least one of the following conditions holds*

(i) there is $1 \leq i \leq n-1$ such that α_i is $B_i \sqsubseteq \exists R$ and α_{i+1} is $\exists R^- \sqsubseteq B'_{i+1}$, where $R \in \Sigma(\mathcal{T})$.

(ii) $B'_n = \exists R$ and $B = \exists R^-$, where $R \in \Sigma(\mathcal{T})$.

Proof. We prove it by induction on n. Assume $n = 1$, thus, g_1 instantiates B'_1 and B, and therefore B'_1 and B share the predicate symbol (concept or role name). Case (i) is not applicable, since there is only one α_i. Assume that Case (ii) does not hold. Then, one of the following options holds: *(1)* either B'_1 is (of the form) $\exists R$ and B is $\exists R$ or *(2)* B'_1 is A and B is A. From either case we conclude that $B_1 \sqsubseteq \neg B' \in \mathsf{cl}(\{\alpha_1, \beta\})$, which contradicts the assumption of the proposition.

Assume $n > 1$. If Case (i) does not hold, then consider α_i and α_{i+1} for some $i < n$. Since g_i instantiates B'_i and B_{i+1}, they share the predicate symbol, thus one of the following options hold: *(1)* either B'_i is $\exists R$ and B_{i+1} is $\exists R$ or *(2)* B'_i is A and B_{i+1} is A. Thus, as in the case above, we conclude that $B_i \sqsubseteq B'_{i+1} \in \mathsf{cl}(\{\alpha_i, \alpha_i + 1\})$. Applying this argument iteratively to pairs of α_i and α_{i+1} from $i = 1, \ldots, n-1$, we obtain that $B_1 \sqsubseteq B'_n \in \{\alpha_1, \ldots, \alpha_n\}$. If Case (ii) does not hold, then, using the same argument as in the case of $n = 1$ to $B_1 \sqsubseteq B'_n$ and $B \sqsubseteq B'$, we obtain that $B_1 \sqsubseteq \neg B' \in \{\alpha_1, \ldots, \alpha_n, \beta\}$, which contradicts the assumption of the proposition. □

Proposition A.4. *Let $\mathcal{K} = (\mathcal{T} \cup \mathcal{A})$ and $\mathcal{N}$ be a simple evolution setting, $\mathcal{I} \models \mathcal{K}$ and $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathbf{L}^a_\subseteq$. If $\mathcal{N} \models_\mathcal{T} \exists R(a)$, $\mathcal{N} \not\models_\mathcal{T} R(a, b)$, and there is an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ such that $\mathcal{I} \cup \{R(a, b)\} \not\models \alpha$, then $R(a, b) \notin \mathcal{J}$.*

Proof. Assume $R(a, b) \in \mathcal{J}$, then consider $\mathcal{J}' = \mathcal{J} \setminus \{R(a, b)\}$. We now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, thus contradicting the fact that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathbf{L}^a_\subseteq$. Since $\mathcal{N} \not\models_\mathcal{T} R(a, b)$, it clearly holds that $\mathcal{J}' \models \mathcal{N}$. Since $\mathcal{N} \models_\mathcal{T} \exists R(a)$ and the evolution setting is simple, there is c such that $\mathcal{N} \models_\mathcal{T} R(a, c)$ Thus, $R(a, c) \in \mathcal{J}'$ and therefore $\mathcal{J}'$ satisfies PIs of $\mathsf{cl}(\mathcal{T})$. Since $\mathcal{J}' \subseteq \mathcal{J}$ and $\mathcal{J}$ satisfies NIs of $\mathsf{cl}(\mathcal{T})$, so does $\mathcal{J}'$. We conclude that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$.

Finally, observe that the fact that $\mathcal{I} \cup \{R(a, b)\} \not\models \alpha$ implies that $R(a, b) \notin \mathcal{I}$. Taking into account that $R(a, b) \notin \mathcal{J}'$, we conclude that $R(a, b) \notin \mathcal{I} \ominus \mathcal{J}'$. At the same time $R(a, b) \in \mathcal{J}$ and therefore $R(a, b) \in \mathcal{I} \ominus \mathcal{J}$. Thus, $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$ holds and we conclude the proof.

□

Proof of Lemma 3.20.
Let $\mathcal{K} = (\mathcal{T} \cup \mathcal{A})$ and $\mathcal{S} = \mathsf{Align}_\mathcal{T}((\mathcal{I} \setminus \mathcal{B}_\mathcal{I}), \mathcal{N})$. Assume there exists a model $\mathcal{J}' \in \mathcal{I} \diamond \mathcal{N}$ under $\mathbf{L}^a_\subseteq$ such that $\mathcal{S} \not\subseteq \mathcal{J}'$. Consider $\mathcal{J}'' = \mathcal{J}' \cup \mathcal{S}$. We will show that $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}'' \subsetneq \mathcal{I} \ominus \mathcal{J}'$ which yields a contradiction with $\mathcal{J}' \in \mathcal{I} \diamond \mathcal{N}$ under $\mathbf{L}^a_\subseteq$.

To see that $\mathcal{I} \ominus \mathcal{J}'' \subsetneq \mathcal{I} \ominus \mathcal{J}'$ observe the following:

$$\begin{aligned}
\mathcal{I} \ominus \mathcal{J}'' &= (\mathcal{I} \setminus \mathcal{J}'') \cup (\mathcal{J}'' \setminus \mathcal{I}) \\
&= ((\mathcal{I} \setminus \mathcal{J}') \setminus \mathcal{S}) \cup \big((\mathcal{J}' \setminus \mathcal{I}) \cup (\mathcal{S} \setminus \mathcal{I})\big) \\
&\subsetneq_{\text{ due to } \mathcal{S} \subseteq \mathcal{I} \text{ and } \mathcal{S} \not\subseteq \mathcal{J}'} (\mathcal{I} \setminus \mathcal{J}') \cup (\mathcal{J}' \setminus \mathcal{I}) \cup (\mathcal{S} \setminus \mathcal{I}) \\
&=_{\text{ due to } \mathcal{S} \subseteq \mathcal{I}} (\mathcal{I} \setminus \mathcal{J}') \cup (\mathcal{J}' \setminus \mathcal{I}) = \mathcal{I} \ominus \mathcal{J}'.
\end{aligned}$$

It remains to show that $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$. The fact that $\mathcal{J}'' \models \mathcal{N}$ follows trivially from the fact that $\mathcal{J}' \in \mathsf{Mod}(\mathcal{N})$, $\mathcal{J}' \subseteq \mathcal{J}''$ and that each assertion of $\mathcal{N}$ is of the form $R(a,b)$ or $A(c)$. We now prove that $\mathcal{J}'' \models \mathcal{T}$ by showing that both $\mathcal{S}$ and $\mathcal{J}'$ are models of $\mathcal{T}$ and then by applying Proposition 3.2. Obviously, $\mathcal{J}' \models \mathcal{T}$ holds by the definition of $\mathcal{J}'$. Observe that by the definition of alignment:

$$\mathcal{S} = (\mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}) \setminus \bigcup_{g \in \mathcal{I}\setminus\mathcal{B}_{\mathcal{I}} \text{ s.t. } \{g\}\cup\mathcal{N} \models_{\mathcal{T}} \bot} \mathsf{root}_{\mathcal{T}}(g).$$

Since $\mathcal{I} \models \mathcal{T}$, one can show that $\mathcal{S} \models \mathcal{T}$ by applying Proposition 3.3 a necessary (probably infinite) number of times: first to $\mathcal{B}_{\mathcal{I}}$ and then to each $g \in \mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}$ s.t. $\{g\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$.

Since we proved that $\mathcal{S} \models \mathcal{T}$ and $\mathcal{J}' \models \mathcal{T}$ we can apply Proposition 3.2, that is, $\mathcal{S} \cup \mathcal{J}' \models \mathcal{T}$ if for every $f \in \mathcal{S}$ and $g \in \mathcal{J}'$ it holds: $\{f, g\} \not\models_{\mathcal{T}} \bot$. Assume this is not the case, and there are $f \in \mathcal{S}$ and $g \in \mathcal{J}'$ such that $\{f, g\} \models_{\mathcal{T}} \bot$. Let $\mathcal{G}$ (resp. $\mathcal{H}$) be the set of atoms g of $\mathcal{J}'$ such that $\{f, g\} \models_{\mathcal{T}} \bot$ for some $f \in \mathcal{S}$ and $\mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g) \cap \mathcal{N} = \emptyset$ (resp. $\mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g) \cap \mathcal{N} \neq \emptyset$). By our assumption, $\mathcal{H} \cup \mathcal{G} \neq \emptyset$. Note that it is enough to consider only the case when f is a unary atom. Indeed, if f is binary, i.e. if $f = R(a,b)$, then, due to the fact that in *DL-Lite*$_{\mathcal{R}}$ disjointness is allowed between basic concepts only, $\{R(a,b), g\} \models_{\mathcal{T}} \bot$ holds if and only if either $\{\exists R(a), g\} \models_{\mathcal{T}} \bot$ or $\{\exists R^-(b), g\} \models_{\mathcal{T}} \bot$. If the first case holds, then we can introduce a fresh concept name $A_{\exists R}$ and extend $\mathcal{I}$ by assigning $A^{\mathcal{I}}_{\exists R} = (\exists R)^{\mathcal{I}}$ to be the interpretation of $R^{\mathcal{I}}$ projected on the fist coordinate. Then, both the original $\mathcal{I}$ and the extended one will behave equivalently wrt to the proposition.

We first show that $\mathcal{H} = \emptyset$. Assume this is not the case and there is $g \in \mathcal{H}$. Let $g' \in \mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g) \cap \mathcal{N}$. By the definition of $\mathsf{root}_{\mathcal{T}}$ for models, there is a sequence of PIs $\alpha_1 = B_1 \sqsubseteq B'_1, \dots, \alpha_n = B_n \sqsubseteq B'_n$ in $\mathsf{cl}(\mathcal{T})$ and of atoms $g_0, \dots, g_n$ in $\mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g)$ such that $g_0 = g'$, $g_n = g$ and $g_{i-1} \to g_i$ for $1 \leq i \leq n$ instantiates α_i. From $\{f, g\} \models_{\mathcal{T}} \bot$ and Lemma 12 of [11] it follows that there is an NI $\beta = B \sqsubseteq \neg B'$ in $\mathsf{cl}(\mathcal{T})$ such that $g_n \to \neg f$ instantiates β. We now show that $B_1 \sqsubseteq \neg B' \notin \mathsf{cl}(\{\alpha_1, \dots, \alpha_n\})$ holds and then apply Proposition A.3. Assume $B_1 \sqsubseteq \neg B' \in \mathsf{cl}(\{\alpha_1, \dots, \alpha_n\})$, then it holds that $\mathcal{T} \models B_1 \sqsubseteq \neg B'$. Moreover, $g_0 \to \neg f$ instantiates $B_1 \sqsubseteq \neg B'$ and therefore, $\{f, g_0\} \models_{\{B_1 \sqsubseteq \neg B'\}} \bot$. Combining $\{f, g_0\} \models_{\{B_1 \sqsubseteq \neg B'\}} \bot$ and $\mathcal{T} \models B_1 \sqsubseteq \neg B'$, we conclude that $\{f, g_0\} \models_{\mathcal{T}} \bot$. Taking into account that $g_0 \in \mathcal{N}$, we finally conclude that $\{f\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. Therefore, $f \notin \mathcal{S}$ which gives a contradiction with $f \in \mathcal{S}$. Thus, we can apply Proposition A.3.

Assume that Case (ii) of Proposition A.3 holds, that is $\alpha_n = B_n \sqsubseteq \exists R$ and $\beta = \exists R^- \sqsubseteq \neg B'$. In particular, this means that g is of the form $R(x,y)$. Since the evolution setting is simple, $\mathcal{T} \not\models \exists R_1 \sqsubseteq \exists R$ for any role R_1. Combining this with $\alpha_n = B_n \sqsubseteq \exists R$, we obtain that B_n and all B_i and B'_i occurring in α_i for $1 \leq i \leq n-1$ are atomic concepts, say A_i and A'_i, respectively. Indeed, let B'_k be of the form $\exists R_1^-$ and has the highest index among B_is with this property. Then, $\mathcal{T} \models \exists R_1^- \sqsubseteq \exists R$, which contradiction the fact that $\mathcal{K}$, $\mathcal{N}$ is a simple setting. This implies that $\{\alpha_1, \dots, \alpha_n\} \models_{\mathcal{T}} A_1 \sqsubseteq \exists R$. Combining this with the our assumption that g' instantiates A_1 and $g = R(x,y)$, we obtain that $g' = A(x)$ and $A_1(x) \to R(x,y)$ instantiates $A_1 \sqsubseteq \exists R$. Since $g' \in \mathcal{N}$, we conclude that $\mathcal{N} \models_{\mathcal{T}} \exists R(x)$. Recall that $\{f, R(x,y)\} \models_{\mathcal{T}} \bot$ and $f \in \mathcal{I}$ (since $f \in \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{I}$), thus $\mathcal{I} \cup \{R(x,y)\}$ does not satisfy at least one NI of $\mathsf{cl}(\mathcal{T})$. Now if $\mathcal{N} \not\models_{\mathcal{T}} R(x,y)$ holds, then we are in the conditions of Proposition A.4 and can conclude that $R(x,y) \notin \mathcal{J}'$, which contradicts the fact that $R(x,y) = g \in \mathcal{J}'$. Therefore, $\mathcal{N} \models_{\mathcal{T}} R(x,y)$. Combining this with $\{f, R(x,y)\} \models_{\mathcal{T}} \bot$, we obtain that $\{f\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. Since f is unary we conclude that $f \in \mathcal{B}_{\mathcal{I}}$ which contradicts the fact that $f \in \mathcal{S}$.

Assume that Case (i) of Proposition A.3 holds but Case (ii) does not. Then, let k be the maximal index satisfying that α_k and α_{k+1} are respectively of the form $B_k \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq B'_{k+1}$. If $k = n-1$, then $\alpha_n = \exists R^- \sqsubseteq B'_n$. Moreover, since Case (ii) of Proposition A.3 does *not* hold and in the evolution settings the entailment $\mathcal{T} \models \exists R^- \sqsubseteq \exists R'$ is not possible for any role R', we have that $B_i = A_i$ and $B'_i = A'_i$ (in fact, it even holds that $B_i = A_i$ and $B'_i = A_{i+1}$) for $1 \leq i \leq n-2$, $B_{n-1} = A_{n-1}$, $B'_n = A$, $B = A$ and $B' = A'$, where all A_j, A'_j and A, A' are from $\Sigma(\mathcal{T} \cup \mathcal{N})$. Thus,

$$\alpha_{n-1} = A_{n-1} \sqsubseteq \exists R, \quad \alpha_n = \exists R^- \sqsubseteq A, \quad g_0 = A_1(x), \quad g_{n-1} = R(x,y), \quad g_n = A(y) \text{ and } f = A'(y)$$

If $\mathcal{N} \models_{\mathcal{T}} R(x,y)$, then $\mathcal{N} \models_{\mathcal{T}} A'(y)$, and we obtain that $\{f\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$, which contradicts $f \in \mathcal{S}$. If $\mathcal{N} \not\models_{\mathcal{T}} R(x,y)$, then due to $R(x,y) \models_{\{\alpha_n\}} A(y)$ and the fact that $\{A(y), A'(y)\}$ violates β, we conclude that $\{R(x,y), A'(y)\}$ violates $\exists R^- \sqsubseteq \neg A' \in \mathsf{cl}(\mathcal{T})$. Thus, $\mathcal{I} \cup \{R(x,y)\}$ violates $\exists R^- \sqsubseteq \neg A'$ and we can apply Proposition A.4 to conclude that that $R(x,y) \notin \mathcal{J}'$, which contradicts the fact that $R(x,y) \in \mathsf{root}_{\mathcal{T}}^{\mathcal{J}'}(g)$. If $1 \leq k < n-1$, then analogously to the previous case we can show that $B'_k = \exists R\ B_{k+1} = \exists R^-$, for each $1 \leq i \leq k-2$ and $k+1 \leq i \leq n$ it holds that $B_i = A_i$ and $B'_i = A'_i$ (in fact, it even holds that $B'_i = A_{i+1}$) and also $B = A'_n$ and $B' = A'$, where all A_j, A'_j, A' and R are from $\Sigma(\mathcal{T} \cup \mathcal{N})$. Moreover,

$$\alpha_k = A_k \sqsubseteq \exists R, \quad \alpha_{k+1} = \exists R^- \sqsubseteq A_{k+2}, \quad g_0 = A_1(x), \quad g_k = R(x,y), \quad g_n = A_n(y) \text{ and } f = A'(y).$$

Thus, applying the same reasoning as above we obtain a contradiction either with $f \in \mathcal{S}$ or $g \in \mathcal{J}'$. We conclude that $\mathcal{H} = \emptyset$ and $\mathcal{G} \neq \emptyset$.

Now consider

$$\hat{\mathcal{J}}' = \mathcal{J}' \setminus \bigcup_{g \in \mathcal{G}} \mathsf{root}_{\mathcal{T}}(g) \cup \bigcup_{h \in \mathsf{root}_{\mathcal{T}}^{\mathcal{J}'}(g) \text{ s.t. } g \in \mathcal{G}, h \in \mathcal{S}} \mathcal{S}[h].$$

We now show that $\hat{\mathcal{J}}' \ominus \mathcal{I} \subsetneq \mathcal{J}' \ominus \mathcal{I}$ and $\hat{\mathcal{J}}' \models \mathcal{T} \cup \mathcal{N}$, which contradicts the fact that $\mathcal{J}' \in \mathcal{I} \diamond \mathcal{N}$ under $\mathbf{L}^a_{\subseteq}$. The inclusion $\hat{\mathcal{J}}' \ominus \mathcal{I} \subseteq \mathcal{J}' \ominus \mathcal{I}$ follows from the fact that each $\mathcal{S}[h] \subseteq \mathcal{I}$. The inclusion is strict since $\mathcal{G} \neq \emptyset$, $\mathcal{G} \subseteq \mathcal{J}'$, and $\mathcal{G} \cap \mathcal{I} = \emptyset$. Since for each $g \in \mathcal{G}$ it holds that $\mathsf{root}_{\mathcal{T}}^{\mathcal{J}'}(g) \cap \mathcal{N} = \emptyset$, we have that $\hat{\mathcal{J}}' \models \mathcal{N}$. To see that $\hat{\mathcal{J}}' \models \mathcal{T}$, observe that $\mathcal{J}' \setminus \cup_{g \in \mathcal{G}} \mathsf{root}_{\mathcal{T}}(g) \models \mathcal{T}$ due to Proposition 3.3, and clearly $\cup_{h \in \mathsf{root}_{\mathcal{T}}^{\mathcal{J}'}(g) \text{ s.t. } h \in \mathcal{S}} \mathcal{S}[h] \models \mathcal{T}$. Therefore, we can apply Proposition 3.2: assume there is $g' \in \mathcal{J}' \setminus \cup_{g \in \mathcal{G}} \mathsf{root}_{\mathcal{T}}(g)$ and $f' \in \mathcal{S}[h]$ for some h such that $\{g', f'\} \models_{\mathcal{T}} \bot$. Since $g' \in \mathcal{J}'$ and $f' \in \mathcal{S}$, one of the two options should hold: either $g' \in \mathcal{G}$ or $g' \in \mathcal{H}$. The first option is impossible since $g' \in \mathcal{J}' \setminus \cup_{g \in \mathcal{G}} \mathsf{root}_{\mathcal{T}}(g)$, and therefore $g' \notin \mathcal{G}$. The second option is also impossible since $\mathcal{H} = \emptyset$. Thus, $\hat{\mathcal{J}}' \models \mathcal{T}$, we obtain a contradiction with $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathbf{L}^a_{\subseteq}$, hence $\mathcal{G} = \emptyset$ and we conclude the proof. □

Proof of Proposition 3.21.
Analogous to the proof of Proposition A.4. □