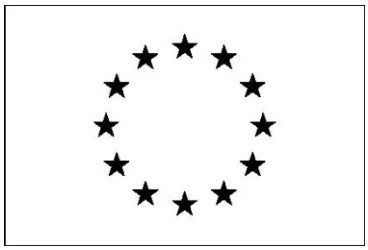

Grant Agreement number: **248095**

Project acronym: **Q-ESSENCE**

Project title: **Quantum Interfaces, Sensors, and Communication based on Entanglement**

Funding Scheme: **Collaborative Project (Large-Scale Integrating Project)**

DELIVERABLE REPORT

Deliverable no.:	**D1.1.2**
Deliverable name:	***Report on analytical bounds for single parameter estimation in open quantum systems***
Workpackage no.	**WP1.1**
Lead beneficiary	**UULM**
Nature	**R = Report**
Dissemination level	**PU = Public**
Delivery date from Annex I (proj month)	**<31/07/2011>**
Actual / forecast delivery date	**<31/07/2011>**
Status	**Submitted**

D1.1.2

Report on analytical bounds for single parameter estimation in open quantum systems

1. Introduction

Parameter estimation in closed systems is Heisenberg-limited, a funtamental bound that can be viewed as a result of the existence of a constrain in the speed of evolution and therefore the time required for states to become optimally distinguishable. When systems are open and in contact with an environment, not many general results are known. Here we discuss two different aspects that shed new light on the limits of parameter estimation in open quantum systems.

2. Concept and Results

Two approaches have been followed:

The first [CHP1-11] analyzes optimal bounds for precision spectroscopy in the presence of general, non-Markovian phase noise. We demonstrate that the metrological equivalence of product and maximally entangled states that holds under Markovian dephasing fails in the non-Markovian case. Using an exactly solvable model of a physically realistic finite band-width dephasing environment, we show that the ensuing non-Markovian dynamics enables quantum correlated states to outperform metrological strategies based on uncorrelated states but otherwise identical resources. We show that this conclusion is a direct result of the coherent dynamics of the global state of the system and environment and, as a result, possesses general validity that goes beyond specific models.

This conclusion holds for both zero and finite temperatures, and is also valid for any other noise model arising from an open quantum system structure. This result can be naturally understood as resulting from the scaling of the optimal interrogation time for entangled particles that is proportional to $n^{-1/2}$ in the number of correlated particles, and therefore causes the entangled probes to experience a suppressed level of decoherence relative to the uncorrelated probes which have to be measured at much longer times. Thus we argue that the result that the relative frequency resolution $r = n^{1/4}$ for rapid measurements is a new metrological limit for entangled particles subject to independent decoherence sources.

In the second approach, lower bounds on the estimation uncertainty are derived for quantum metrological schemes in the presence of decoherence showing that Heisenberg scaling is generically lost even for infinitesimal level of noise [DGK1-12]. Unlike in other methods, calculation of the bounds is straightforward and requires only a simple analysis of the mathematical structure of the decoherence process. In some models, e.g., atomic clocks frequency calibration with dephasing, calculation may be performed using an intuitive geometric picture. All that is necessary is a “distance” of a point representing the decoherence process from the boundary of the set of all quantum channels.

From the examples studied, we may conclude that the classical simulation method may provide a surprisingly tight bound in some decoherence models. The cases, when it fails, are the ones when channels are extremal or ϕ-extremal. Notice, however, that those channels will in general have a simpler structure – fewer linearly independent Kraus operators — as they necessarily lie on the boundary of the set of quantum channels. This gives an opportunity to apply other methods based on the channel extension or minimization over purifications.

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Quantum Metrology in Non-Markovian Environments

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(Dated: March 8, 2011)

We analyze optimal bounds for precision spectroscopy in the presence of general, non-Markovian phase noise. We demonstrate that the metrological equivalence of product and maximally entangled states that holds under Markovian dephasing fails in the non-Markovian case. Using an exactly solvable model of a physically realistic finite band-width dephasing environment, we show that the ensuing non-Markovian dynamics enables quantum correlated states to outperform metrological strategies based on uncorrelated states but otherwise identical resources. We show that this conclusion is a direct result of the coherent dynamics of the global state of the system and environment and, as a result, possesses general validity that goes beyond specific models.

Correlated quantum states can be used to achieve a resolution in metrology that surpasses the precision limits achievable with uncorrelated probes, a significant result of both fundamental and practical relevance first put forward by C. Caves [1]. The potential usefulness of entangled states in overcoming the shot noise limit in precision spectroscopy was proposed in [2], and the first experimental results concerning precision measurements using entangled input states have been presented recently [3]. However, the saturation of the Heisenberg limit by maximally entangled states assumes a fully coherent evolution, whereas in real experiments there will always be some degree of decoherence or a limitation on the total time over which measurements can be performed. Precision spectroscopy in the presence of Markovian decoherence was first analyzed in [4], where it was shown that given a fixed number of particles n and a total available time T for the frequency estimate to be determined, uncorrelated and maximally entangled preparations of n particles achieve exactly the same precision when subject to Markovian dephasing. Hence these two preparations are *metrologically* equivalent in those circumstances.

Here, we analyse if this equivalence persists when the system is subject to non-Markovian noise. Under the same rules as above, namely given n particles and a total time T over which measurements can be performed, we show that in the presence of realistic, finite temperature, finite bandwidth environments, a measurement strategy can always be found in which the use of n-particle entangled states leads to a lower frequency uncertainty when compared to the use of n uncorrelated input states. Moreover, we demonstrate on very general grounds that for these strategies the ratio between the optimal resolution of entangled and uncorrelated probes obeys a characteristic power law $\propto n^{1/4}$. These results imply that entangled states *can* be used to gain an advantage for precision measurements in the presence of noise, and that entanglement-enhanced metrology could be practically implemented in wide variety of condensed matter systems such as realisations of solid-state qubits and biomolecular systems which are typically subject to non-Markovian environments characterized by long correlation times and/or structured spectral features [5].

To show this, let us consider a system Hamiltonian $\omega_0\sigma_z$ which is subject to a system-environment interaction that induces pure dephasing, the form of noise that tends to manifest at the shortest time scales in most qubit realizations. In this case, the coupling to the environment is of the form $\sigma_z \otimes B$, where B is some operator only including bath degrees of freedom. Then, denoting by $(|1\rangle, |0\rangle)$ the eigenbasis of σ_z, quite generically, the time evolution of the reduced density matrix of the system satisfies

$$\rho_{ii}(t) = \rho_{ii}(0) \;\text{ for }\; (i = 0, 1) \tag{1}$$

$$\rho_{01}(t) = \rho_{01}(0)e^{-2\gamma(t)}. \tag{2}$$

Now we consider a typical Ramsey spectroscopy set-up for n uncorrelated particles [6] to find that the resulting single particle signal is given by

$$p_0 = \frac{1}{2}\left(1 + \cos(\Delta t)e^{-\gamma(t)}\right), \tag{3}$$

where Δ is the detuning between the frequency ω of the external oscillator and the atomic frequency ω_0 to which we intend to lock it to and t is the time between Ramsey pulses [4]. Using the same notation as in [4], the best resolution in the estimation of ω_0 is given by the expression

$$\delta\omega_0^2 = \frac{1}{NF(\Delta)}, \tag{4}$$

where N is the total number of experimental data ($N = (T/t)\, n$) and F is the so-called Fisher information [7]. This quantity can be easily evaluated in our case as

$$F(\Delta) = \sum_{i=0,1} \frac{1}{p_i}\left(\frac{\partial p_i}{\partial \Delta}\right)^2. \tag{5}$$

We then find the frequency uncertainty to be

$$\delta\omega_0^2 = \frac{1 - \cos^2(\Delta t)e^{-2\gamma(t)}}{nTt\sin^2(\Delta t)e^{-2\gamma(t)}}. \tag{6}$$

We wish to determine the best operating point Δ and the best interrogation time t_u which minimize Eq. (6),

as these two quantities are under experimental control. To this end, one computes the derivatives of $\delta\omega_0^2$ with respect to Δ and t and then equates these derivatives with 0. Independent of the choice of $\gamma(t)$, we conclude from the derivative of $\delta\omega_0^2$ with respect to Δ that

$$\Delta t_u = \frac{k\pi}{2} \tag{7}$$

for odd k or, in other words, the choice that ensures $\cos\Delta t_u = 0$ is optimal. Inserting $\Delta t_u = \frac{k\pi}{2}$ in the expression for the derivative with respect to t_u to eliminate Δ, these expressions simplify considerably and we obtain the second constraint

$$2t\frac{d\gamma}{dt}(t)|_{t=t_u} = 1. \tag{8}$$

Using the Eq. (7) in Eq. (6) we have

$$\delta\omega_0^2|_{t_u} = \frac{1}{nTt_u}e^{2\gamma(t_u)}, \tag{9}$$

where the optimal interrogation time t_u is determined by Eq. (8). The Markovian case is recovered from these equations for $\gamma(t) = \gamma(0)\,t$ and the expressions above reduce to those presented in [4].

An analogous calculation can be done for an initial preparation of n particles in a maximally entangled state $|0\rangle^{\otimes n} + |1\rangle^{\otimes n}$, leading to the result that the optimal frequency resolution is

$$\delta\omega_0^2|_e = \frac{1}{n^2Tt_e}e^{2n\gamma(t_e)}, \tag{10}$$

where the optimal interrogation time for *entangled* particles t_e is determined by the constraint,

$$2nt_e\frac{d\gamma}{dt}(t)|_e = 1. \tag{11}$$

In the Markovian case, the additional factor of n in the denominator of Eq. (10) is cancelled out due to an accompanying decrease in t_e by a factor of n relative to t_u. The optimal frequency resolution is therefore identical to that obtained with uncorrelated particles, and thus maximally entangled and uncorrelated states are *metrologically* equivalent in the presence of Markovian dephasing. Although Markovian dephasing does not allow any advantage to be gained from using maximally entangled states, the conclusions drawn above are very general, as the expressions above do not depend on the precise form of the decoherence model. Provided that it generates Markovian dephasing, the bath operator B could be highly non-linear, complexly structured, quantum or classical.

We now move beyond the standard Markovian treatment and consider the performance of maximally entangled states in the presence of non-Markovian dephasing. We shall first study some specific, exactly solvable models, which demonstrate that entangled and uncorrelated probes are no longer metrologically equivalent in the presence of non-Markovian dynamics, and then discuss why this result is in fact independent of the microscopic details of the environment for most realistic system-bath structures.

An exactly solvable model – Let us first consider the exactly solvable model (independent boson model) [8]. Here the bath operator B is simply a sum of linear couplings to the coordinates of a continuum of harmonic oscillators described by a spectral function $J(\omega)$ [8–10]. Then we have

$$\gamma(t) = \frac{1}{2}\int_0^\infty d\omega J(\omega)\coth\left(\frac{\omega\beta}{2}\right)\frac{1-\cos(\omega t)}{\omega^2}. \tag{12}$$

where β is the inverse temperature.

Power-law spectral densities with exponential cut-offs – The coupling to a bath of harmonic oscillators is the most common setting used in the study of open-quantum systems, and an extremely large number of physical environments can be described by a general power-law form for the spectral density [9, 10]. Following Ref. [8], we therefore consider $J(\omega) = \alpha\omega_c^{1-s}\omega^s e^{-\omega/\omega_c}$, where α is a dimensionless coupling constant and ω_c cuts off the spectral density at high frequencies. For zero temperature, $t > 0$ and $s > 0$, we obtain the result

$$\gamma(t) = \frac{\alpha}{2}\left(1 - \frac{\cos[(s-1)\tan^{-1}(\omega_c t)]\,\Gamma(s-1)}{(1+\omega_c^2t^2)^{\frac{s-1}{2}}}\right) \tag{13}$$

where $\Gamma(s-1)$ is the Euler Gamma function. Taking the limit $s \to 1$ carefully, one also finds

$$\gamma(t, s=1) = \frac{a}{2}\ln(1+\omega_c^2t^2). \tag{14}$$

From Eq. (13) one immediately sees that at short ($\omega_c t \ll 1$) and long ($\omega_c t \gg 1$) times, $\gamma(t)$ has a power law dependence on time, and it is therefore instructive to analyse a generic $\gamma(t)$ of the from $\gamma(t) = \alpha t^\nu$. We define the relative frequency resolution of entangled and uncorrelated probes $r = |\delta\omega_0|_u/|\delta\omega_0|_e$. We then find

$$r^2 = n\left(\frac{t_e}{t_u}\right)e^{2\gamma(t_u)-2n\gamma(t_e)}. \tag{15}$$

In the absence of dephasing noise, $r = \sqrt{n}$ (Heisenberg limit), while in the Markovian case the metrological equivalence of the correlated and entangled probes is presented by the result $r = 1$. Using the constraint equations Eq. (7) and (8), it can be seen that for the general power law form of $\gamma(t) = \alpha t^\nu$, we always obtain $\gamma(t_u) = n\gamma(t_E)$ and the exponential term in Eq. (15) always equals unity. Hence r is determined by the ratio of best interrogation times t_u/t_e. Similarly, from Eqs. (8) and (11) one can show that the ratio $t_u/t_e = n^{1/\nu}$ and therefore $r^2 = n^{\frac{\nu-1}{\nu}}$. From this result we see that only for $\nu > 1$ is there an advantage to using entangled

probes, and r approaches the Heisenberg limit from below as $\nu \to \infty$. The case of $\nu = 1$ corresponds to the Markovian case, whilst $\nu < 1$ always favours uncorrelated probes.

With this analysis we can use Eq. (13) to assess r as a function of the bath exponents s. For short times, expanding Eq. (13) to the leading-order in $\omega_c t$, it can be seen that for all spectral densities $\gamma(t) \propto t^2$, and one then obtains $r = n^{\frac{1}{4}}$. The necessary interrogation times for entangled states scale like $t_e \propto (\omega_c \sqrt{n})^{-1}$, which is consistent with the short time approximation of $\gamma(t)$. In many cases, and particularly in molecular and magnetic systems, the conditions on the measurement time may be met easily with current experimental methods due to the sluggishness of the dephasing environments. We also note that in the limit of a static bath which induces Gaussian inhomogeneous broadening, $\gamma(t) \propto t^2$ even for long times [11].

For times much greater than ω_c^{-1}, we find that $\gamma(t) \propto t^{1-s}$ for $0 < s < 1$. For this case, known as sub-Ohmic dissipation [9, 10], uncorrelated probes are always favoured, while for $s = 1$ we can analytically evaluate the optimal interrogation times for each initial preparation without considering the long or short time limits. The exact result is

$$r = \sqrt{n} f(\alpha, n), \tag{16}$$

where

$$f(\alpha, n) = \sqrt{\left[\frac{(2\alpha/(2\alpha-1))^{\alpha}}{(2n\alpha/(2n\alpha-1))^{n\alpha}}\right]\sqrt{2\alpha - 1/2n\alpha - 1}}, \tag{17}$$

and $\alpha > 1/2$ [12]. The results are shown in Figure 1, illustrating that maximally entangled states in the presence of zero temperature Ohmic baths outperform uncorrelated probes for any n, with $r \to n^{\frac{1}{4}}$ as $n \to \infty$ and/or $\alpha \to \infty$.

Lorentzian spectral density – Now we consider the spectral density

$$J(\omega) = \frac{1}{\pi}\frac{ag}{g^2 + \omega^2},$$

where a regulates the coupling strength. We then find for $T = 0$ that

$$\gamma(t) = \frac{a}{4g}\left(\frac{1}{g}(e^{-gt} - 1) + t\right) \tag{18}$$

for $g \geq 0$ and $t \geq 0$. Now inserting the necessary condition $\Delta t = \pi/2$ in the expression for $\delta\omega_0^2$ we obtain

$$\delta\omega_0^2|_u = \frac{1}{nTt} e^{\frac{a(-1+e^{-gt}+gt)}{2g^2}}.$$

The second necessary condition for an optimum imposes that the optimal time satisfies

$$at(1 - e^{-gt}) = 2g.$$

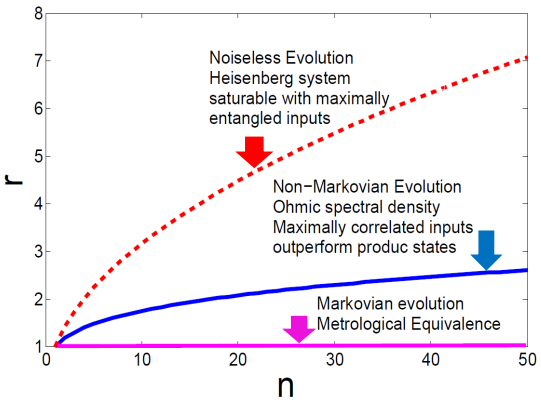

FIG. 1. Ratio r between the optimal resolution achievable with uncorrelated and maximally entangled inputs as a function of the number of particles n. The dashed line shows the expected behaviour in the absence of noise where $r = \sqrt{n}$ (Heisenberg limit), while $r = 1$ (pink line) when the noise is Markovian. In the presence of non-Markovian phase decoherence, product states and maximally entangled initial preparations are no longer metrologically equivalent. In the case of a zero temperature bath with an Ohmic spectral density ($s = 1$), maximally entangled states allow for a higher resolution for any value of n and r displays a typical $n^{1/4}$ dependence as shown by the solid line in the figure.

This is a transcendental equation but, if we are interesting in the short time behaviour $gt \ll 1$, then we find in lowest order as an approximate solution

$$t^2 = \frac{2}{a}$$

and employing the Newton method on the function $f(t) = at(1 - e^{-gt}) - 2g$ with starting point $t = \sqrt{\frac{2}{a}}$ we find the improved value

$$t = \sqrt{\frac{2}{a}}\left(1 + \sqrt{\frac{g^2}{8a}}\right).$$

Inserting this into the expression for $\delta\omega_0^2$ we find

$$\delta\omega_{opt}^2|_u = \frac{1}{nT}\sqrt{\frac{a}{2}}\frac{\sqrt{8a}}{\sqrt{8a} + g} e^{\frac{g}{3}\sqrt{\frac{2}{a}} - 1}$$

Repeating the calculation for a maximally entangled state, we obtain in the short time limit $gt \ll 1$,

$$\delta\omega_{opt}^2|_e = \frac{1}{nT}\sqrt{\frac{a}{2n}}\frac{\sqrt{8an}}{\sqrt{8an} + g} e^{\frac{g}{3}\sqrt{\frac{2}{an}} - 1}.$$

We find an improved precision for maximally entangled states as $\delta\omega_{opt}^2$ is reduced by a factor $\sqrt{n}$ whenever $8an \gg g^2$. If that last condition is not satisfied, the above approximate expressions fail to hold, as then g becomes large. A numerical calculation reveals that for $8an \ll g^2$, maximally entangled and product states

achieve the same precision and the optimal interrogation time becomes large. That entangled and product states then achieve the same precision can be expected as memory effects in the bath become negligible for large interrogation times.

Beyond specific models – The key point illustrated by the examples above is that maximally entangled states achieve their optimal interrogation time at shorter time intervals than uncorrelated states and can hence benefit more from non-Markovian noise features. This is due to the characteristic behaviour $\gamma(t) \propto t^2$ which governs short times in the models above, and which leads to a decrease in the optimal interrogation time for entangled particles that only scales as $n^{-\frac{1}{2}}$ (c.f. $t_e \propto n^{-1}$ for the Markovian case).

However, the quadratic behaviour of $\gamma(t)$ is not a specific feature of our chosen noise model, but rather a general consequence of the unitary evolution of the total system and environment state. The essential observation is that the function $\gamma(t)$ appears in the dynamics of the sub-system as the result of transitions induced in the bath by the system-bath interaction. At a short time t after the system-bath interaction is switched on, the probability for the bath state to make a transition to any state orthogonal to its initial condition is *always* proportional to t^2. This universal time dependence for quantum mechanical transitions is the fundamental basis of the quantum Zeno effect, and has been extensively and rigourously investigated [13, 14]. Therefore, for essentially *all* noise sources treated within the standard framework of open-quantum system theory, entangled-state input protocols can always be found which outperform uncorrelated probes, whatever the microscopic details of the bath and the system-bath interaction.

This general result leads to the concept of a new fundamental limit for quantum metrology in the presence of noise, which for simplicity we shall refer as the *Zeno limit*. For sufficiently fast interrogation times, we find the model-independent scaling law for the Zeno limit $r = n^{\frac{1}{4}}$, which is below the Heisenberg limit $r = \sqrt{n}$, but always above the Markov limit $r = 1$. For the specific noise models studied above, we also find that t_e can be simply related to r through the relation $r^2\omega_c t_e = 1$ at $T = 0$ K. Again, given the universal scaling law for r, a relation of this form should also be expected to hold for other noise models, except that ω_c should be replaced by the fastest dynamical frequency of the environment in these models.

Finite Temperatures – The arguments given above also naturally apply to the case of finite temperatures, where again we find that a Zeno-limit emerges. However, the typical energy scale that determines the optimal interrogation time t_e now depends explicitly on temperature. This can be seen explicitly in the high temperature limit of our exact model, where the factor $\coth(\beta\omega_c/2)$ in Eq. (13) can be expanded to leading order in $\beta\omega_c$ over the entire integration range. For an Ohmic bath this leads to $\gamma(t) = \alpha\beta^{-1}(t\tan^{-1}(\omega_c t) - \ln(\sqrt{1+\omega_c^2 t^2})\,\omega_c^{-1})$. Again, a Zeno-limit appears at short times with $\gamma(t) \approx \alpha\beta^{-1}\omega_c t^2/2$, which leads to the result $t_E = \sqrt{\frac{\beta}{4\alpha n\omega_c}}$. This result, derived in the high temperature limit, is consistent with our notion that it is the fastest time scale of the bath dynamics that sets the scale for the Zeno-limit interrogation time. If the system is interrogated slower than this timescale, we find that entangled and uncorrelated probes become equivalent again as $\gamma(t) \approx \alpha\beta^{-1}\omega_c t$ at long times and the Markov result is recovered.

Conclusions – Using an exactly soluble model of non-Markovian dephasing we have shown that entangled probes can outperform uncorrelated probes provided the system is interrogated on time scales faster than the characteristic frequencies of the bath dynamics. This conclusion holds for both zero and finite temperatures, and is also valid for any other noise model arising from an open-quantum system structure. This result can be naturally understood as resulting from the scaling $t_e \propto n^{-1/2}$ in the number of correlated particles, which causes the entangled probes to experience a suppressed level of decoherence relative to the uncorrelated probes which have to be measured at much longer times. Thus we argue that the result $r = n^{1/4}$ for rapid measurements is a new metrological limit for entangled particles subject to independent decoherence sources.

Acknowledgements – Financial support from the EU Integrated Project Q-Essence, STREP action CORNER and an Alexander von Humboldt Professorship is gratefully acknowledged.

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Quantum enhanced metrology and the geometry of quantum channels

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Lower bounds on the estimation uncertainty are derived for quantum metrological schemes in the presence of decoherence showing that Heisenberg scaling is generically lost even for infinitesimal level of noise. Unlike in other methods, calculation of the bounds is straightforward and requires only a simple analysis of the mathematical structure of the decoherence process. In some models, e.g. atomic clocks frequency calibration with dephasing, calculation may be performed using an intuitive geometric picture. All that is necessary is a "distance" of a point representing the decoherence process from the boundary of the set of all quantum channels.

PACS numbers: 06.20.Dk, 03.65.Vf, 03.65.Yz, 42.50.St

I. INTRODUCTION

Since the recognition of the fact that entangled states allow for sensing with precision, which scales beyond the limits achievable with classically correlated probes, quantum enhanced metrology has been a field of extensive research both theoretically [1–4] and experimentally [5–20]. Parameter estimation with N independent probes, as a consequence of the central limit theorem, yields the, $1/\sqrt{N}$, *standard scaling* (SS) of precision, in which case the optimal asymptotic estimation schemes are known [21]. In optical interferometry, where N is the number of photons used, the SS is more commonly referred to as the shot noise scaling. Entangling the probes, however, can in principle offer a quadratic precision enhancement, i.e. the, $1/N$, *Heisenberg scaling* (HS) [22–25]. This potential boost is due to the high sensitivity of entangled states to the variations of the estimated parameter. Such strategies have been experimentally realized in optical interferometry [5–10] with spectacular application in the quest of the first direct detection of gravitational waves, where, by the usage of squeezed light, sub-shot noise sensitivity has already been achieved [11, 12]. Moreover, the same quantum enhancement principle can be utilized in atomic spectroscopy [26, 27]. Recent experiments have proved that an improved frequency standards in atomic clocks can be established with atoms prepared in special entangled states — the spin squeezed states [13–18].

Unfortunately, both the theory [28–33] and experiments [20] confirmed the fragility of the above mentioned systems to the unavoidable noise - *decoherence*. Furthermore, in the asymptotic limit of infinite resources, $N \to \infty$, even infinitesimally small noise most likely turns HS into SS, so that the quantum gain amounts then to a constant factor improvement. This behavior has been shown in the case of optical *interferometry with photonic losses* [34, 35], and in the case of atomic spectroscopy with noise modeled as *dephasing* [36], by means of universally valid SS bounds' derivations.

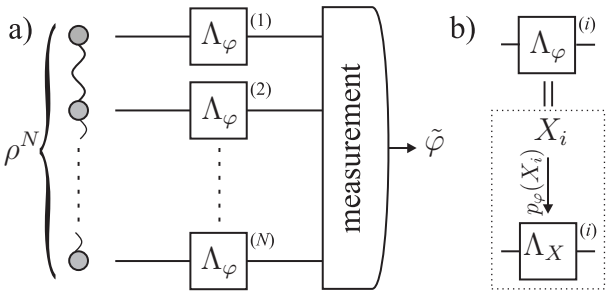

Figure 1. a) General scheme for quantum enhanced metrology. N-probe quantum state fed into N parallel channels is sensing an unknown channel parameter φ. An estimator $\tilde{\varphi}$ is inferred from a measurement results on the output state. b) Classical simulation of a quantum channel. The channel Λ_φ is interpreted as a mixture of other channels Λ_X, where the dependence on φ is moved into the mixing probabilities $p_\varphi(X)$.

In this paper, we show that the SS bounds can be derived for nearly all types of decoherence models by analyzing the structure of a single use of the decoherence channel, regardless of both the choice of input states and the measurement strategies of estimation. We make use of the necessary *Fujiwara-Imai condition* (FIC) [37] as well as the beautifully simple concept of *classical simulation* (CS) [38], in order to derive SS bounds for important quantum metrological models with decoherence. While in [37, 38] it has been proven that SS bounds can be derived for full rank channels, i.e. the ones that do not lie at

the boundary of the set of all physical channels, we show that the methods may be applied to a more general class including the ones lying on the boundary of the set. This allows us to calculate SS bounds in a simple way, sometimes making use solely of the geometry of quantum channels, for the most important decoherence models in quantum metrology such as: dephasing, depolarization, spontaneous emission and losses inside an interferometer. Moreover, we relate the CS to the FIC as well as compare the derived SS bounds with the ones obtained previously in the literature using systematic, but much more computationally demanding, methods such as [36].

II. BOUNDS ON PRECISION IN QUANTUM ENHANCED METROLOGY

The most general evolution of a quantum system may be described via its purification — a unitary interaction with an environment and subsequent tracing out of the environmental degrees of freedom. Provided the system is initially in state ρ_S, decoupled from the environment E, the evolution can always be written as

$$\begin{aligned}\Lambda_\varphi[\rho_S] &= \mathrm{Tr}_E\left\{U_\varphi^{SE}\left(\rho_S\otimes|\zeta\rangle_E\langle\zeta|\right)U_\varphi^{SE\dagger}\right\}\\ &= \sum_i K_i(\varphi)\,\rho_S\,K_i^\dagger(\varphi)\,,\end{aligned} \tag{1}$$

where φ is the evolution parameter, $K_i(\varphi) = {}_E\langle\xi_i|U_\varphi^{SE}|\zeta\rangle_E$ are the Kraus operators with vectors $\{|\xi_i\rangle\}_i$ chosen to be any that form a complete basis in the environment's Hilbert space. Clearly, as this choice is ambiguous, the dynamics or the quantum channel can be represented equivalently by different sets of Kraus operators that are related to one another via

$$\forall_i:\quad \tilde{K}_i(\varphi) = \sum_j \mathbf{u}_{ij}(\varphi)\,K_j(\varphi) \tag{2}$$

with $\mathbf{u}(\varphi)$ being a unitary matrix, which entries could in principle depend on the evolution parameter φ. Additionally, for a quantum channel to be trace preserving, Kraus operators need to satisfy $\sum_i K_i^\dagger(\varphi)K_i(\varphi)=\mathbb{1}$ [39].

In quantum metrology one is concerned with the task of estimating with highest precision the evolution parameter, φ, of a quantum channel, Λ_φ, acting in parallel on an N-probe system, i.e. $\Lambda_\varphi^{\otimes N}[\rho^N]$ (see Fig. 1a). In order to do so, one has not only to find the optimal (possibly highly entangled) input state, ρ^N, but also the most effective measurement strategy on the output, which will minimize the estimation's uncertainty $\Delta\varphi_N$. Although the optimal strategy may be found for the small scale problems, the optimization becomes intractable, when the number of probes increases. In this regime one is forced to look for efficiently calculable lower bounds on the estimation uncertainty. The bounds reported in this paper are directly determinable, as they are based only on the structure of a *single use* of a quantum channel Λ_φ, disregarding the N-fold tensor product $\Lambda_\varphi^{\otimes N}$ and the properties of N-probe input states.

The problem of φ-estimation may be pursued in the so called *local approach*. This is the most optimistic case, when one is searching for a strategy that offers highest sensitivity to deviations from an *a priori* known φ_0. Then, the ultimate precision can be quantified without the need of optimization over the measurement stage, but only the input state, ρ^N, by bounding the minimal uncertainty as [24]

$$\Delta\varphi_N \geq \frac{1}{\sqrt{\mathcal{F}_N}},\ \mathcal{F}_N = \max_{\rho^N} F_Q\left[\Lambda_\varphi^{\otimes N}\left[\rho^N\right]\right], \tag{3}$$

where the *quantum Fisher information* (QFI), F_Q, of a state ϱ_φ is given by $F_Q[\varrho_\varphi] = \mathrm{Tr}\{\varrho\mathcal{L}_\varphi^2\}$. The operator $\mathcal{L}_\varphi$ is the *symmetric logarithmic derivative*, implicitly defined by the relation $\partial_\varphi\varrho_\varphi = \frac{1}{2}\left[\mathcal{L}_\varphi\varrho_\varphi+\varrho_\varphi\mathcal{L}_\varphi\right]$. For pure states the formula for F_Q simplifies to

$$F_Q(|\psi_\varphi\rangle) = 4(\langle\dot{\psi}_\varphi|\dot{\psi}_\varphi\rangle - |\langle\dot{\psi}_\varphi|\psi_\varphi\rangle|^2),\ |\dot{\psi}_\varphi\rangle = \partial_\varphi|\psi_\varphi\rangle. \tag{4}$$

The states that maximize the QFI and yield the HS for decoherence free unitary channels are highly entangled ones, such as the GHZ state in the case of the atomic spectroscopy [40], and the N00N state in the case of the optical interferometry [41]. In the presence of decoherence, the structure of optimal input states is no more intuitive and the maximization of the QFI (with rising N) becomes hard even numerically [27–29]. In order to study the asymptotic behavior, one may be better off employing not necessarily tight, but analytically calculable, upper bounds on the QFI.

A. Minimization over channel purifications

A way of constructing such bounds is to derive and employ other equivalent definitions of the QFI. It has been shown in [36, 37] that

$$F_Q\left[\Lambda_\varphi\left[\rho\right]\right] = \min_{|\Psi_\varphi\rangle} F_Q(|\Psi_\varphi\rangle), \tag{5}$$

where the minimization is performed over all φ differentiable purifications of the output state, $\Lambda_\varphi[\rho] =$

$\mathrm{Tr}_E\{|\Psi_\varphi\rangle\langle\Psi_\varphi|\}$. For pure input, only the channel must be purified and the purifications then correspond to its different equivalent Kraus representations, as in Eq. (2).

In this case, by an educated guess of a channel purification, one can derive excellent analytical bounds, which with some effort can be made input independent and give asymptotic scaling of precisions for many quantum metrological models [36]. Nevertheless, the method may be cumbersome, when the channel description involves many Kraus operators, hence, many possible purifications and additionally requires some analysis of the properties of the N-probe input state.

B. Channel extension

A simpler, although not necessarily saturable, bound can be derived exploiting an intuitive observation that, allowing the channel to act in a trivial way on an extended space, can only improve the estimation's precision, i.e. $\max_\rho F_Q\left[\Lambda_\varphi\left[\rho\right]\right] \leq \max_{\rho_{\mathrm{ext}}} F_Q\left[\Lambda_\varphi \otimes \mathbb{I}\left[\rho_{\mathrm{ext}}\right]\right]$. This leads to an upper bound on $\mathcal{F}_N$, which goes around the input state optimization, yielding [37]:

$$\mathcal{F}_N \leq 4 \min_K \left\{ N \left\|\alpha_K\right\| + N\left(N-1\right)\left\|\beta_K\right\|^2 \right\} \quad (6)$$

with $\alpha_K = \sum_i \dot{K}_i^\dagger(\varphi)\dot{K}_i(\varphi)$, $\beta_K = \mathrm{i}\sum_i \dot{K}_i^\dagger(\varphi)K_i(\varphi)$, $\dot{K}_i(\varphi) = \partial_\varphi K_i(\varphi)$, $\|\cdot\|$ denoting the operator norm, and the minimization being performed over all equivalent Kraus representations of Λ_φ [see Eq. (2)].

Clearly, from the above inequality we can prove the SS of precision, if we are able to find a Kraus representation of the channel that satisfies $\beta_K = 0$, since then the $\mathcal{F}_N$ will scale asymptotically at most linearly with N. Equivalently, this amounts to finding a Hermitian matrix $\mathbf{h}$, i.e. the generator of the unitary Kraus rotation, $\mathbf{u}(\varphi) = \exp[-\mathrm{i}\mathbf{h}\varphi]$, of Eq. (2), that satisfies the *Fujiwara-Imai condition* (FIC) [37]

$$\sum_{i,j} \mathbf{h}_{ij} K_i^\dagger K_j = \mathrm{i}\sum_q \dot{K}_q^\dagger K_q. \quad (7)$$

Moreover, if this is possible we can then easily derive a quantitative SS bound:

$$\mathcal{F}_N \leq 4N \min_{\mathbf{h}} \left\|\alpha_{\tilde{K}}\right\|, \quad (8)$$

where $\tilde{K}_i = \sum_j [\exp(-i\mathbf{h}\varphi)]_{ij} K_j$.

The FIC greatly narrows the search for the optimal Kraus representation and therefore it distinguishes a class of Kraus rotation matrices, $\mathbf{u}(\varphi)$, which can be also efficiently used in the previously mentioned method of [36]. Nevertheless, the problem of the minimization may still be challenging in some cases. The method presented in the next subsection does not have this drawback and provides a clear geometrical interpretation of the results.

C. Classical simulation

The overall process of estimation, depicted in Fig. 1a, can be represented by a Markov chain,

$$\varphi \to \Lambda_\varphi^{\otimes N}\left[\rho^N\right] \to \tilde{\varphi}, \quad (9)$$

meaning that the true value φ determines the channel Λ_φ that is then used N times in parallel to produce the output state, on which a measurement is performed, from which an estimate of the parameter $\tilde{\varphi}$ is constructed.

Let us consider a *classical simulation* [38] of the channel, Λ_φ, i.e. a probabilistic mixture of some other physical channels (see Fig. 1b):

$$\Lambda_\varphi[\rho] = \int dX\, p_\varphi(X)\, \Lambda_X[\rho]. \quad (10)$$

Crucially, the estimated parameter specifies only the probability distribution, p_φ, of some random, possibly multidimensional variable, X, that indicates, which channel to pick from the φ-independent set $\{\Lambda_X\}$. Making use of N independent random variables X_i distributed according to $p_\varphi(X_i)$, we can rewrite the estimation process as

$$\varphi \to p_\varphi \to \{X_i\}_{i=1}^N \to \otimes_{i=1}^N \Lambda_{X_i}\left[\rho^N\right] \to \tilde{\varphi}, \quad (11)$$

which, by the Markov property, must be equivalent or worse than the estimation process

$$\varphi \to p_\varphi \to \{X_i\}_{i=1}^N \to \tilde{\varphi}. \quad (12)$$

This description, however, corresponds just to the classical estimation using N independent and identically distributed variables. Hence, using the classical Cramï¿œr-Rao bound (CRB) we obtain a lower bound on the uncertainty of the original problem (9)

$$\Delta\varphi_N \geq \frac{1}{\sqrt{N\, F_{\mathrm{cl}}[p_\varphi]}},\ F_{\mathrm{cl}} = \int dX \frac{[\partial_\varphi p_\varphi(X)]^2}{p_\varphi(X)}. \quad (13)$$

For an alternative derivation of the above result see Appendix A.

Provided the F_{cl} is finite and $p_\varphi(X)$ satisfies regularity conditions required in the derivation of CRB

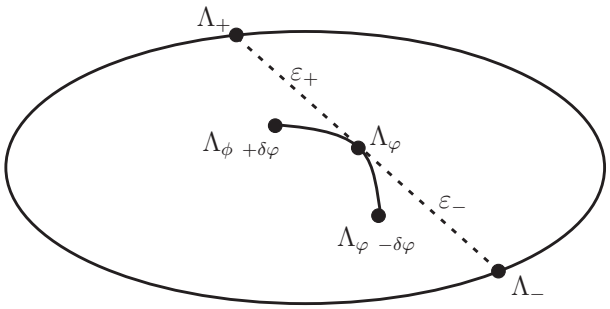

Figure 2. Schematic representation of a local classical simulation of a channel that lies *inside* the convex set of quantum channels.

[42], one can immediately infer that only SS of precision is possible for the problem considered. Moreover, choosing a particular classical simulation one may derive an explicit SS bound on precision. If a channel admits many classical simulations, i.e. different p_φ, then we obtain the tightest SS bound for $\Delta\varphi_N$ by choosing the "worst" decomposition yielding minimal $F_{\rm cl}[p_\varphi]$.

HS is possible only when the above mentioned conditions are *not* satisfied. This is clearly the case in the decoherence-free situation, when Λ_φ is unitary. Unitary operations are extremal elements of the set of quantum channels, and therefore the only admissible p_φ is the non-regular Dirac delta distribution being zero on all channels except Λ_φ. For such probability distribution $F_{\rm cl}=\infty$, and therefore the previous reasoning does not imply the SS of precision.

Recall that QFI at a given value φ_0 is dependent only on the output state and its first derivative at φ_0. This implies that any channel, $\tilde{\Lambda}_\varphi$, which "locally coincides" with the original one, i.e. $\forall_\rho\colon\ \tilde{\Lambda}_{\varphi_0}[\rho]=\Lambda_{\varphi_0}[\rho],\ \partial_\varphi\tilde{\Lambda}_\varphi[\rho]\Big|_{\varphi_0}=\partial_\varphi\Lambda_\varphi[\rho]|_{\varphi_0}$, must also satisfy $F_Q[\Lambda_\varphi^{\otimes N}[\rho]]\big|_{\varphi_0}=F_Q[\tilde{\Lambda}_\varphi^{\otimes N}[\rho]]\Big|_{\varphi_0}$. It is therefore enough to consider the *local classical simulations*, i.e. any mixtures reproducing the channel and its first derivative at given φ_0. As mentioned earlier, the most "informative" classical simulation are the ones yielding the smallest $F_{\rm cl}$. As proven in Appendix B, these can be constructed using two channels $\{\Lambda_+,\Lambda_-\}$, that lie on the tangent line to the "channel trajectory" at the two outermost points situated on the border of all the physical maps, see Fig 2. The local classical simulation reads explicitly

$$\tilde{\Lambda}_\varphi[\rho]=p_\varphi^+\,\Lambda_+[\rho]+p_\varphi^-\,\Lambda_-[\rho], \tag{14}$$

with $\Lambda_\pm[\rho]=\Lambda_{\varphi_0}[\rho]\pm\varepsilon_\pm\partial_\varphi\Lambda_\varphi[\rho]\big|_{\varphi_0}$ and $p_\varphi^\pm=\frac{\varepsilon_\mp\pm(\varphi-\varphi_0)}{\varepsilon_++\varepsilon_-}$ chosen, so that $\tilde{\Lambda}_\varphi$ "locally coincides" with Λ_φ at φ_0. Making use of Eq. (13) applied to binary probability distribution $p_\varphi^\pm$ we obtain

$$\Delta\varphi_N\geq\sqrt{\frac{\varepsilon_+\varepsilon_-}{N}}. \tag{15}$$

All that is needed to calculate the above bound are the "distances" $\varepsilon_\pm$ of the channel from the boundary measured along the tangent line.

The above bound is useless for extremal channels since then $\varepsilon\pm=0$. If a channel is non-extremal, then the above construction will yield a finite $F_{\rm cl}$ provided that $\varepsilon_\pm>0$, i.e. it can be decomposed into mixtures of channels lying on the tangent. It could be that, although the channel is non-extremal, it is non-extremal "in the wrong direction" and $\varepsilon_\pm=0$ — we call such channels *φ-extremal*. All channels that are *not* φ-extremal manifest the SS of precision. In particular, all full rank channels (i.e. channels not lying on the boundary of the set of quantum channels) are limited by the SS of precision [37, 38].

Geometry of the channels and more specifically the φ-extremality are best viewed with the use of the Choi-Jamiołkowski isomorphism [43, 44]. Given a quantum channel $\Lambda\colon\mathcal{L}(\mathcal{H}_{\rm in})\mapsto\mathcal{L}(\mathcal{H}_{\rm out})$ acting from the space of density matrices on $\mathcal{H}_{\rm in}$ to density matrices on $\mathcal{H}_{\rm out}$, one defines $P_\Lambda=\Lambda\otimes\mathbb{I}\,[|\Phi\rangle\langle\Phi|]$, where $|\Phi\rangle=\sum_{i=1}^{\dim\mathcal{H}_{\rm in}}|i\rangle\otimes|i\rangle$ is a maximally entangled state in $\mathcal{H}_{\rm in}\otimes\mathcal{H}_{\rm in}$. Λ is a physical channel (i.e. trace preserving, completely positive map) iff P_Λ is a positive semi-definite operator, satisfying $\mathrm{Tr}_{\mathcal{H}_{\rm out}}\{P_\Lambda\}=\mathbb{1}$. If $\{K_i\}_i$ are the Kraus operators of the Λ channel, we can write explicitly $P_\Lambda=\sum_i|K_i\rangle\langle K_i|$, where $|K_i\rangle=K_i\otimes\mathbb{1}|\Phi\rangle$.

We can now say that, the channel Λ_φ is *not* φ-extremal, if it is possible to find a non-zero ε, for which $P_{\Lambda_\varphi}\pm\varepsilon\,\partial_\varphi P_{\Lambda_\varphi}\geq 0$. See Appendix C for an alternative formulation of the φ-extremality condition and its relation to the well known non-extremality condition due to Choi [44]. In practice, if we want to make most out of the bound in Eq. (15), we need to find the maximum values of $\varepsilon_\pm$, for which $P_{\Lambda_\varphi}\pm\varepsilon_\pm\,\partial_\varphi P_{\Lambda_\varphi}$ are still positive semi-definite operators. This is a simple eigenvalue problem and therefore the bound can be obtained immediately.

Finally, we note that any channel, which can be locally simulated, satisfies also the FIC [proof in Appendix D] which also implies that the CS bound will never be tighter than the one resulting from finding the global minimum in Eq. (8). This extends the space of FIC fulfilling channels from full rank [37], i.e. lying inside the set of quantum maps, to ones being φ non-extremal. Note, however, that FIC can

also be potentially satisfied by some extremal channels for which CS is not possible, e.g. the qubit amplitude damping channel discussed in the next section.

III. EXAMPLES AND COMPARISON OF THE METHODS

All examples of channels presented below are of the form $\Lambda_\varphi[\rho] = \Lambda\big[U_\varphi \rho U_\varphi^\dagger\big]$, i.e. a concatenation of a unitary rotation encoding the estimate parameter φ and an φ-independent decoherence process. Consequently, $K_i(\varphi) = K_i U_\varphi$, where $K_i(\varphi)$, K_i are the Kraus operators of Λ_φ and Λ respectively.

In what follows, we adopt the standard notation where $\mathbb{1}$ is the 2×2 identity matrix and $\{\sigma_i\}_{i=1}^3$ are the Pauli operators. We focus on two-level probe systems sensing a phase shift modeled using a unitary $U(\varphi) = \exp[(\mathrm{i}\sigma_3\varphi)/2]$. In the case of atomic clocks' frequency calibration $\varphi = \delta\omega \cdot t$ with $\delta\omega$ being the detuning between the frequency of the atomic transition and the frequency of driving field, while t the time of evolution. Even though, in practice, it is the $\delta\omega$ that is the parameter to be estimated, we will still consider φ as the estimation parameter, in order to have a unified notation for atomic and optical models. In the case of a two mode optical interferometer, $U(\varphi)$ is the operator acting on a single photon state, accounting for the accumulated relative phase shift, φ, between the two arms of the interferometer.

A. Dephasing

Dephasing of a qubit is a typical decoherence model applied to two level atoms subject to fluctuating external magnetic/laser fields. The canonical Kraus operators read [39]

$$K_1 = \sqrt{1-p}\,\mathbb{1},\; K_2 = \sqrt{p}\,\sigma_3, \tag{16}$$

where $0 \le p \le 1/2$ is the dephasing strength. The Choi-Jamiołkowski isomorphism of the corresponding Λ_φ channel $P_{\Lambda_\varphi} = \sum_{i=1}^2 |K_i U_\varphi\rangle\langle K_i U_\varphi|$ has a simple form:

$$P_{\Lambda_\varphi} = \begin{pmatrix} 1 & 0 & 0 & \mathrm{e}^{\mathrm{i}\varphi}(1-2p) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{e}^{-\mathrm{i}\varphi}(1-2p) & 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

It is easy to check that $P_{\Lambda_\varphi} + \varepsilon_\pm \partial P_{\Lambda_\varphi} \ge 0$ provided $\epsilon_\pm \le (2\sqrt{p(1-p)})/(1-2p)$, hence using Eq. (15) we arrive at the bound given in Table I.

channel considered	**classical simulation** (15)	**channel extension** (8)	**channel purification** (5)
dephasing	$\frac{2\sqrt{(1-p)p}}{1-2p}$	$\frac{2\sqrt{(1-p)p}}{1-2p}$	
depolarisation	$\frac{\sqrt{p(4-3p)}}{2(1-p)}$	$\frac{\sqrt{p(4-3p)}}{2(1-p)}$	
amplitude damping	N/A	$\frac{1}{2}\sqrt{\frac{p}{1-p}}$	
lossy interferometer	N/A	$\sqrt{\frac{1-\eta}{\eta}}$	

Table I. Precision bounds of the most relevant models in quantum enhanced metrology derived with three different methods. All bounds are of the form $\Delta\varphi_N \ge \frac{\text{const}}{\sqrt{N}}$, where constant factors are given in the table. Classical simulations do not provide any useful bounds for amplitude damping and lossy interferometer, since these channels are φ-extremal.

The only situation, in which the bound does not provide any information is when $p = 0$, what is not surprising, since then the channel is unitary — i.e. extremal. Most importantly, the above bound, is exactly the same as the one derived using the channel extension method. The global minimum in Eq. (8) can be found explicitly and corresponds to $\mathbf{h} = \sigma_1/[4\sqrt{p(1-p)}]$. Furthermore, $\mathbf{h}$ is exactly the generator leading to the purification employed in [36], where, after dealing with the unavoidable optimization over input, same result was derived — see Table I.

This proves, the usefulness of the CS solution, which with a minimal effort allows to derive bounds that are equally tight as the ones derived with much more advanced methods. Notice that we have derived the bound, even though the channel is *not* full rank (lies on the boundary of the set).

B. Depolarization

Qubit depolarization of strength $0 \le p \le 1$ defined via the Kraus operators

$$K_1 = \sqrt{1-\frac{3p}{4}}\,\mathbb{1},\; \left\{K_i = \sqrt{\frac{p}{4}}\,\sigma_i\right\}_{i=1\ldots3} \tag{18}$$

is an example of a full rank channel (for $p > 0$), hence we can automatically infer the SS of precision.

Analogously to the dephasing case, a bound using the CS can be simply derived and we include it in Table I. We were not able to derive a better bound neither from (Eq. (8)) nor by minimization over purifications, which is again a hint on the power of the CS method. It might be that with the more advanced methods it is possible to improve the bound, yet it will surely require much more effort.

Next, in order to provide reader with the full picture, we describe two decoherence models, in which the CS method is not applicable.

C. Amplitude damping (spontaneous emission)

The well known two-level atom spontaneous emission model is described by the Kraus operators

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \; K_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \tag{19}$$

with $0 \leq p \leq 1$. Interestingly, for all p this channel is extremal [45]. Because of the extremality, the CS is not helpful to derive bounds on precision. Nevertheless, the channel extension method of Eq. (8), can be easily employed.

Substituting $K_i(\varphi)$ into Eq.(7) we find that the FIC fixes the generator to $\mathbf{h} = \frac{1}{2p} \begin{pmatrix} p & 0 \\ 0 & p-2 \end{pmatrix}$. Consequently, there is no need for minimization over the Kraus decompositions and from Eq. (8) we obtain an SS bound listed in Table I.

Moreover, using $\mathbf{h}$ as the generator of the purification's rotation and optimizing over the input according to procedure of [36], we arrive at exactly same bound. However, taking into account that in the channel extension method we did not have to perform any kind of optimization, the first result remains impressive.

D. Lossy interferometer

In order to model *loss in an optical interferometer*, a third orthogonal state at the output — vacuum — resulting from a loss of a photon needs to be introduced. The decoherence channel on a single probe (single photon) is a map from a two to a three dimensional system:

$$K_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \sqrt{1-\eta} \end{pmatrix}, \; K_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \sqrt{1-\eta} & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & \sqrt{\eta} \\ 0 & 0 \end{pmatrix}, \tag{20}$$

where η is the power transmission coefficients for the light traveling through each of the two arms [46]. Although the corresponding channel Λ_φ is non-extremal, it is unfortunately φ-extremal and again CS cannot be used. Still, similarly as in the amplitude damping case the FIC condition is satisfied. Restricting ourselves to diagonal generators $\mathbf{h}$ the optimal bound (see Table I) corresponds to the choice $\mathbf{h}_{11} = 0$, $\mathbf{h}_{22} = \frac{\eta}{1-\eta}$, $\mathbf{h}_{33} = -\frac{1}{2}$. Not surprisingly, this bound equals the one obtained in [34–36], because $\mathbf{h}$ turned out to be the same generator as the one used in [36] of the optimal purification.

IV. SUMMARY

From the examples studied, we may conclude that the CS method may provide a surprisingly tight bounds in some decoherence models. The cases, when it fails, are the ones when channels are extremal or φ-extremal. Notice, however, that extremal or φ-extremal channels will in general have a simpler structure – fewer linearly independent Kraus operators — as they necessarily lie on the boundary of the set of quantum channels. This gives an opportunity to apply other methods based on the channel extension or minimization over purifications. The general rule of thumb might be that, the further from extremal, φ-extremal points the channel is, the more useful is the CS method to derive precision bounds. From a practical point of view one might say that, since there are always many sources of noise present, a typical realistic decoherence model will almost certainly be described by a quantum channel well inside the set, hence far from the set's boundaries. In such a case, the CS is a perfect tool to get an immediate lower bound on the achievable precision.

ACKNOWLEDGMENTS

We thank Konrad Banaszek for many inspiring discussions and constant support. This research was founded by by the European Commission under the IP project Q-ESSENCE, ERA-NET CHIST-ERA

project QUASAR and the Foundation for Polish Science under the TEAM programme.

Appendix A: Proof of the classical simulation bound, Eq. (13)

In order to simplify the reasoning, let us focus on the classical simulations that are constructed using discrete sets of quantum channels, $\{\Lambda_i\}$. Then, the considered channel's action can be rewritten as a φ-independent map acting on a larger input space [38]

$$\Lambda_\varphi[\rho] = \sum_i p_{\varphi,i}\,\Lambda_i[\rho] = \Phi\left[\rho \otimes \sigma_\varphi\right], \tag{A1}$$

where $\sigma_\varphi = \sum_i p_{\varphi,i}\,|e_i\rangle\langle e_i|$ represents a diagonal state in a basis, in which Φ is defined via $\Phi[\varrho] = \sum_i \Lambda_i \otimes P_i\,[\varrho]$ with $P_i[\sigma] = \langle e_i|\,\sigma\,|e_i\rangle$. In order to prove (13), we write the QFI for the N parallel use of the channel and bound it from above, i.e.

$$\begin{aligned} F_Q\big[\Lambda_\varphi^{\otimes N}[\rho]\big] &= F_Q\big[\Phi^{\otimes N}\left[\rho \otimes \sigma_\varphi^{\otimes N}\right]\big] \\ &\leq F_Q\big[\rho \otimes \sigma_\varphi^{\otimes N}\big] = F_Q\big[\sigma_\varphi^{\otimes N}\big] \\ &= N\,F_Q[\sigma_\varphi] = N\,F_{\rm cl}[p_\varphi], \end{aligned} \tag{A2}$$

exploiting the monotonicity of the QFI under any parameter independent quantum map, here $\Phi^{\otimes N}$, [47].

Appendix B: Optimal Tangent Simulation

We prove that the optimality of CS is reached by choosing two channels, which are as far as possible from Λ_{φ_0} on the tangent, but still lie in the set of physical maps. Consider a CS being a mixture composed of channels that lie on the tangent line to the channel trajectory. Pick a channel Λ_0 situated a position x_0, $\Lambda_0 = \Lambda_{\varphi_0} + x_0\,\partial_\varphi \Lambda_\varphi|_{\varphi_0}$, which contributes to the mixture with p^0_φ. Let us now remove this channel from the mixture and replace it by two channels $\Lambda_\pm$ at positions $x_\pm$, $x_- < x_0 < x_+$, with mixing probabilities $p^\pm_\varphi$. For the total probability to sum up to one and the new mixture to be locally equivalent to the previous one, the following quantities as well as their first derivatives over φ should equalize at φ_0:

$$p^+_\varphi + p^-_\varphi = p^0_\varphi, \quad p^+_\varphi\,x_+ + p^-_\varphi\,x_- = p^0_\varphi\,x_0. \tag{B1}$$

Solving the above set of equations we get

$$p^\pm_{\varphi_0} = p^0_{\varphi_0}\frac{x_\pm - x_0}{x_\pm - x_\mp}, \quad \partial_\varphi p^\pm_\varphi|_{\varphi_0} = \partial_\varphi p^0_\varphi|\varphi_0 \frac{x_\pm - x_0}{x_\pm - x_\mp}. \tag{B2}$$

We now want to show that the described operation can only decrease the Fisher information $F_{\rm cl}$. For a moment consider a situation in which channels $\Lambda_\pm$ were not used in the original mixture and only appeared at the expense of the removed Λ_0 channel. The change of $F_{\rm cl}$ is given by

$$\delta F_{\rm cl} = \frac{[\partial_\varphi p^+_\varphi]^2}{p^+_\varphi} + \frac{[\partial_\varphi p^-_\varphi]^2}{p^-_\varphi} - \frac{[\partial_\varphi p^0_\varphi]^2}{p^0_\varphi}\bigg|_{\varphi=\varphi_0} \tag{B3}$$

and by Eq. (B2) it must be zero.

If, on the other hand, channels $\Lambda_\pm$ were already in use in the original mixture, we can artificially separate their initial part from the one originated at the expense of taking away probability from the point x_0. This formally corresponds to introducing additional distinguishing symbol that would split the instances of the classical random variable $x_\pm$ into two groups, depending on their origin. By the same reasoning as before, the $F_{\rm cl}$ must remain constant during probabilities redistribution. Finally, removing the distinguishing symbol we can only reduce the $F_{\rm cl}$, what proves that indeed replacing Λ_0 with $\Lambda_\pm$ in a local classical simulation can only decrease the $F_{\rm cl}$.

This reasoning may now be applied to an arbitrary probability distribution of channels lying on the tangent line. We can only decrease the $F_{\rm cl}$ by deleting one of the channels and redistributing its probability to the ones lying at the line ends. Repeating this procedure for all channels on the tangent line we conclude that the optimal (yielding minimum $F_{\rm cl}$) simulation is the one composed of two channels situated at the intersection of the tangent line with the boundary of the set of all quantum channels.

The missing point in the above reasoning is to prove that it is not possible to decrease $F_{\rm cl}$ further by including channels that lie *outside* the tangent line. We have proved this fact for the class of channels $\Lambda_\varphi[\rho] = \Lambda(U_\varphi \rho U^\dagger_\varphi)$ — concatenation of a unitary parameter encoding and a decoherence map. The sketch of the proof is the following. Notice that the values of parameters x_i appearing in Eq. (B2) will now be obtained by performing a projection of the channel onto the tangent line. The only way, in which the use of this larger class of channels could lead to lower $F_{\rm cl}$, would be the situation when after projecting a channel onto the tangent line, we find ourselves outside the physical region of all quantum maps. The effective x_i lies then further apart than when considering only channels on the tangent line. This seems like a possible way to increase the product $\varepsilon_+\varepsilon_-$. However, taking one channel, e.g. Λ_+, to the "top" of the tangent line requires taking the complementary channel, i.e. Λ_-, to the "bottom". Since

the decoherence structure is φ-invariant and the set of channels is convex, the gain in the ε_+ will be compensated by at least an equal loss in the distance ε_-, which will *not* make the product $\varepsilon_+\varepsilon-$ larger, and therefore will *not* decrease the $F_{\rm cl}$.

Appendix C: φ non-extremality

We prove below that the condition on φ non-extremality, which requires the existence of a non-zero epsilon such that

$$P_{\Lambda_\varphi} \pm \varepsilon\, \partial_\varphi P_{\Lambda_\varphi} \geq 0, \tag{C1}$$

is equivalent to the statement that there exist a non zero Hermitian matrix μ_{ij} such that

$$\partial_\varphi P_{\Lambda_\varphi} = \sum_{ij} \mu_{ij}\, |K_i\rangle\langle K_j|\,. \tag{C2}$$

Assume that (C1) holds and recall that $P_{\Lambda_\varphi} = \sum_i |K_i\rangle\langle K_i|$. Since the operator on the l.h.s. is positive

$$\langle\psi|\left[\sum_i |K_i\rangle\langle K_i| \pm \varepsilon \sum_i |\dot{K}_i\rangle\langle K_i| + |K_i\rangle\langle\dot{K}_i|\right]|\psi\rangle \geq 0 \tag{C3}$$

for any vector $|\psi\rangle$. In order to prove (C2), it is enough to show that $|\dot{K}_j\rangle$ can be written as linear combinations of K_i. If this was not the case for one of vectors, e.g. $|\dot{K}_{\bar{i}}\rangle$, we would additionally need a vector $|L_{\bar{i}}\rangle$ that is orthogonal to the the space spanned by $\{|K_i\rangle\}_i$, then taking $|\psi\rangle = \sqrt{p}|K_{\bar{i}}\rangle + \exp(i\xi)\sqrt{1-p}|L_{\bar{i}}\rangle$ positivity condition (C3) leads to

$$p|\langle K_{\bar{i}}|K_{\bar{i}}\rangle|^2 \pm \sqrt{p(1-p)}\varepsilon\left(e^{\mathrm{i}\xi}\langle L_{\bar{i}}|\dot{K}_{\bar{i}}\rangle + c.c\right) \geq 0 \tag{C4}$$

For any nonzero ε and nonzero $\langle L_{\bar{i}}|\dot{K}_{\bar{i}}\rangle$ we can always find some ξ and p small enough, so that the l.h.s. is negative. This leads to a contradiction, hence (C2) must hold.

For the opposite direction, assume (C2) holds and substitute the formula for $\partial P_{\Lambda_\varphi}$ into the l.h.s. of (C2). We need to show that there exist ε such that

$$A^\pm = \sum_i |K_i\rangle\langle K_i| \pm \varepsilon \sum_{ij} \mu_{ij} |K_i\rangle\langle K_j| \geq 0. \tag{C5}$$

Defining matrices $\nu^\pm = \mathbf{1} \pm \varepsilon\mu$, note that, for ε small enough, they are positive semi-definite. Hence, we can take their square root and construct $|\tilde{K}_i^\pm\rangle = \sum_j \left[\sqrt{\nu^\pm}\right]_{ij} |K_j\rangle$. Then, $A^\pm = \sum_i |\tilde{K}_i\rangle\langle\tilde{K}_i|$, which is clearly positive semi-definite.■

As a last remark, observe that tracing out (C2) over $\mathcal{H}_{\rm out}$ space yields

$$0 = \sum_{ij} \mu_{ij} K_i^\dagger K_j, \tag{C6}$$

which is the Choi condition for non-extremality [44]. This reflects the trivial fact that if a channel is φ non-extremal, then it must *not* be extremal.

Appendix D: Classical simulation implies FIC

If a channel can be locally simulated at φ_0, than up to the first order in $\varphi - \varphi_0$ we have

$$\begin{aligned}\Lambda_\varphi[\rho] &= p_\varphi^+\, \Lambda_+[\rho] + p_\varphi^-\, \Lambda_-[\rho] \\ &= \sum_i p_\varphi^+ K_i^+ \rho K_i^{+\dagger} + \sum_j p_\varphi^- K_j^- \rho K_j^{-\dagger} \\ &= \sum_k \tilde{K}_q \rho \tilde{K}_q^\dagger,\end{aligned}$$

where $\left\{\tilde{K}_q\right\} = \left\{\sqrt{p_\varphi^+}K_i^+\right\} \cup \left\{\sqrt{p_\varphi^-}K_j^-\right\}$. The constructed Kraus operators satisfy FIC, since

$$\beta_{\tilde{K}} = \mathrm{i}\sum_q \dot{\tilde{K}}_q^\dagger \tilde{K}_q = \mathrm{i}\Big(\sum_i \dot{p}_\varphi^+ K_i^{+\dagger} K_i^+ + \tag{D1}$$

$$+ \sum_j \dot{p}_\varphi^- K_j^{-\dagger} K_j^-\Big) = \mathrm{i}\left(\dot{p}_\varphi^+ + \dot{p}_\varphi^-\right)\mathbb{1} = 0, \tag{D2}$$

where we have used $\forall_\varphi\colon\ p_\varphi^+ + p_\varphi^- = 1$.

Moreover, notice that

$$\|\alpha_{\tilde{K}}\| = \left\|\sum_q \dot{\tilde{K}}_q \dot{\tilde{K}}_q\right\| \leq \left(\partial_\varphi\sqrt{p_\varphi^+}\right)^2 \left\|\sum_i K_i^{+\dagger} K_i^+\right\| + \tag{D3}$$

$$\left(\partial_\varphi\sqrt{p_\varphi^-}\right)^2 \left\|\sum_i K_i^{-\dagger} K_i^-\right\| = \frac{[\partial p_\varphi^+]^2}{4p_\varphi^+} + \frac{[\partial p_\varphi^-]^2}{4p_\varphi^-}. \tag{D4}$$

Substituting the above inequality to Eq. (8) we see that $\mathcal{F}_N \leq N F_{\rm cl}$, where $F_{\rm cl}$ is just the Fisher information for the local CS. This shows that the bound derived from CS will never be tighter than the one derived from the channel extension method optimized over all Kraus decompositions.

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