# Final Report for Deliverable Nr. 3.1

## Representation of Simplicial Complexes

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**Abstract**

Simplicial complexes are combinatorial models of topological spaces, which can be used for mechanizing the computation of algebraic invariants. In this report we explain how formalizing them in theorem provers, including their relations with other important concepts in Algebraic Topology as simplicial sets and chain complexes. The aim is oriented towards the applications, namely towards the homological processing of digital images.

## 1 Introduction

Algebraic Topology is a complex and abstract mathematical subject; however, some of its techniques can be applied to different contexts such as coding theory [23], data analysis [12], robotics [16] or digital image analysis [7, 8] (in this last case, in particular in the study of medical images [21]).

Here, we are going to focus on the application of Algebraic Topology to the study of monochromatic digital images.

Let us suppose that we want to study some properties of the image depicted in the left side of Figure 1 by means of Algebraic Topology methods.

From the image depicted in the left side of Figure 1, we can construct a combinatorial object which consists of the triangles, edges and vertices depicted in the right side of Figure 1. This combinatorial object is an Algebraic Topology object called *simplicial complex*.

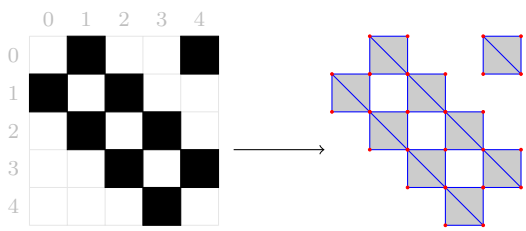


Figure 1: Small 2D-image

$$\begin{tikzcd}
\text{Digital Image} \arrow[r] & \text{Simplicial Complex} \arrow[r] & \text{Simplicial Set} \arrow[r] & \text{Chain Complex} \arrow[r] & \text{Homology} \arrow[llll, bend left, "\text{interpretation}"]
\end{tikzcd}$$

Figure 2: Using Algebraic Topology to study Digital Images

From the simplicial complex, we construct another Algebraic Topology object which is a graded structure, that object is called *simplicial set*. The elements in dimension $0, 1$ and $2$ of the simplicial set are respectively the vertices, the edges and the triangles of the previously constructed simplicial complex. It is worth noting than in dimension higher than 3 there are not any element.

To study the properties of the simplicial set, we need to use one of the main notions of Algebraic Topology, which is that of a *chain complex*. The chain complex associated with the simplicial set allows us to compute the homology groups. In an intuitive sense, homology groups measure "n-dimensional holes" in topological spaces. In this case the result that we obtain is $H_0 = \mathbb{Z} \oplus \mathbb{Z}$ and $H_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Eventually, from these homology groups, we can interpret properties of the original image. Namely, $H_0$ measures the number of connected components of the image that is equal to the number of $\mathbb{Z}$'s in $H_0$, so the image has two connected components; and $H_1$ measures the number of holes of the image that is equal to the number of $\mathbb{Z}$'s in $H_1$, so the image has three holes.

The above process, which is diagramatically described in Figure 2, can be implemented in a software system.

It is worth noting that if we want to apply our method in real life problems (for instance, in the study of medical images), we must be completely sure that the results produced by our programs are correct. Therefore, the formal verification of our programs with a Theorem Prover tool is significant. In particular, we have used both the ACL2 Theorem Prover [14] and the proof assistant Coq [4, 2] as well as the SSReflect extension [6] and the libraries it provides.

# 2 Mathematical Preliminaries

In this section, we briefly provide the minimal mathematical background needed in the rest of the report. We mainly focus on definitions. Many good textbooks are available for these definitions and results about them, the main one being maybe [17].

## 2.1 Digital Images

Let $n$ be any positive integer. An *$n$-xel* $q$ in an Euclidean $n$-space, $\mathbb{R}^n$, is a closed unit $n$-dimensional (hyper)cube $q \subset \mathbb{R}^n$ whose $2^n$ vertices have natural coordinates (more precisely, an *$n$-xel* in $\mathbb{R}^n$ is a cartesian product like $[i_1, i_1 + 1] \times [i_2, i_2 + 1] \times \ldots \times [i_n, i_n + 1]$). In this report, a *pixel* is a 2-xel in $\mathbb{R}^2$. We define an *$n$-dimensional binary image* or *$n$D-image*, to be a finite set of $n$-xels in $\mathbb{R}^n$.

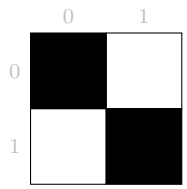


Figure 3: A digital image

An $n$D-image $\mathcal{I}$ can, of course, be represented by a finite $n$-dimensional array of 1's and 0's in which each 1 represents an $n$-xel in $\mathcal{D}$ and each 0 represents an $n$-xel that is not in $\mathcal{D}$.

**Example 1.** Consider the 2D-image depicted in Figure 3. This image can be codified by means of the 2-dimensional array: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

## 2.2 Simplicial Complexes

The most elementary method to settle a connection between general Topology and Algebraic Topology is based on the usage of simplicial complexes. The notion of topological space is too "abstract" in order to perform computations. Simplicial complexes provide a purely combinatorial description of topological spaces which admit a triangulation. The computability of properties, such as homology groups, from a *finite* simplicial complex associated with a topological space is well-known and, for instance in the case of homology groups the algorithm uses simple linear algebra [22]. Then, an algebraic topologist can identify a compact triangulable topological space with a finite simplicial complex, making computations easier.

Let us start with the basic terminology. Let $V$ be an ordered set, called the *vertex set*. An *(ordered abstract) simplex* over $V$ is any ordered finite subset of $V$. An *(ordered abstract) $n$-simplex* over $V$ is a simplex over $V$ whose cardinality is equal to $n+1$. Given a simplex $\alpha$ over $V$, we call *faces* of $\alpha$ to all the subsets of $\alpha$.

**Definition 2.** An *(ordered abstract) simplicial complex* over $V$ is a set of simplexes $\mathcal{K}$ over $V$ such that it is closed by taking faces (subsets); that is to say:

$$\forall \alpha \in \mathcal{K},\ if\ \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K}$$

Let $\mathcal{K}$ be a simplicial complex. Then the set $S_n(\mathcal{K})$ of $n$-simplexes of $\mathcal{K}$ is the set made of the simplexes of cardinality $n+1$ of $\mathcal{K}$.

**Example 3.** Let us consider $V = (0,1,2,3,4,5,6)$.

The small simplicial complex drawn in Figure 4 is mathematically defined as the object:

$$\mathcal{K} = \left\{ \begin{array}{l} \emptyset, (0), (1), (2), (3), (4), (5), (6), \\ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3), (3,4), (4,5), (4,6), (5,6), \\ (0,1,2), (4,5,6) \end{array} \right\}.$$

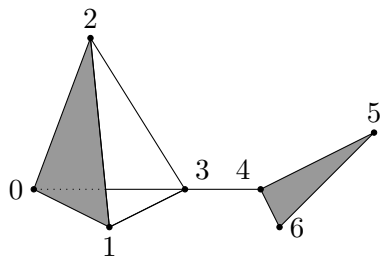


Figure 4: Butterfly Simplicial Complex

Note that, because the vertex set is ordered the list of vertices of a simplex is also ordered, which allows us to use a sequence notation $(\ldots)$ and not a subset notation $\{\ldots\}$ for a simplex and also for the vertex set $V$. It is also worth noting that simplicial complexes can be infinite. For instance if $V = \mathbb{N}$ and the simplicial complex $\mathcal{K}$ is $\{(n)\}_{n\in\mathbb{N}} \cup \{(n-1, n)\}_{n\geq 1}$, the simplicial complex obtained can be seen as an infinite bunch of segments.

**Definition 4.** A *facet* of a simplicial complex $\mathcal{K}$ over $V$ is a maximal simplex with respect to the subset relation, $\subseteq$, among the simplexes of $\mathcal{K}$.

**Example 5.** The facets of the small simplicial complex depicted in Figure 4 are:

$$\{(0,3),(1,3),(2,3),(3,4),(0,1,2),(4,5,6)\}$$

Let us note that a *finite* simplicial complex can be generated from its facets taking the set union of the power set of each one of their facets. In general, we have the following definition.

**Definition 6.** Let $\mathcal{S}$ be a finite sequence of simplexes, then the set union of the power set of each one of the elements of $\mathcal{S}$ is, trivially, a simplicial complex called the *simplicial complex associated with* $\mathcal{S}$.

It is worth noting that the same simplicial complex can be generated from two different sequences of simplexes; in addition, the minimal sequence of simplexes which generates a finite simplicial complex is the sequence of its facets.

## 2.3 Simplicial Sets

In spite of being a powerful tool, many common constructions in Topology are difficult to be made explicit in the framework of simplicial complexes. It soon became clear around 1950 that the notion of simplicial set would be much more suitable.

Simplicial sets were first introduced by Eilenberg and Zilber [5], who called them *semi-simplicial complexes*. They can also be used to express some topological properties of spaces by means of combinatorial notions.

**Definition 7.** A *simplicial set* $K$, is a union $K = \bigcup_{q\geq 0} K^q$, where the $K^q$ are disjoints sets, together with functions:

$$\begin{array}{lll} \partial_i^q : K^q \to K^{q-1}, & q > 0, & i = 0, \ldots, q, \\ \eta_i^q : K^q \to K^{q+1}, & q \geq 0, & i = 0, \ldots, q, \end{array}$$

subject to the relations:

$$\begin{array}{llllll}
(1) & \partial_i^{q-1}\partial_j^q & = & \partial_{j-1}^{q-1}\partial_i^q & \text{if} & i<j, \\
(2) & \eta_i^{q+1}\eta_j^q & = & \eta_j^{q+1}\eta_{i-1}^q & \text{if} & i>j, \\
(3) & \partial_i^{q+1}\eta_j^q & = & \eta_{j-1}^{q-1}\partial_i^q & \text{if} & i<j, \\
(4) & \partial_i^{q+1}\eta_i^q & = & identity & = & \partial_{i+1}^{q+1}\eta_i^q, \\
(5) & \partial_i^{q+1}\eta_j^q & = & \eta_j^{q-1}\partial_{i-1}^q & \text{if} & i>j+1.
\end{array}$$

The $\partial_i^q$ and $\eta_i^q$ are called *face* and *degeneracy* operators respectively.

The elements of $K^q$ are called *q-simplexes*. A simplex $x$ is called *degenerate* if $x = \eta_i y$ for some simplex $y$ and some degeneracy operator $\eta_i$; otherwise $x$ is called *non degenerate*.

## 2.4 Chain Complexes

Now, we are going to introduce a central notion in Algebraic Topology. We assume as known the notions of ring, module over a ring and module morphism (see [13] for details).

**Definition 8.** Given a ring $R$, a *graded module* $M$ is a family of left $R$-modules $(M_n)_{n\in\mathbb{Z}}$.

**Definition 9.** Given a pair of graded modules $M$ and $M'$, a *graded module morphism* $f$ of degree $k$ between them is a family of module morphisms $(f_n)_{n\in\mathbb{Z}}$ such that $f_n : M_n \to M'_{n+k}$ for all $n \in \mathbb{Z}$.

**Definition 10.** Given a graded module $M$, a *differential* $(d_n)_{n\in\mathbb{Z}}$ is a family of module endomorphisms of $M$ of degree $-1$ such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$.

The previous definitions define a graded structure and a way of going from a level of the structure to the inferior one. From the previous definitions, the notion of chain complex is defined as follows.

**Definition 11.** A *chain complex* $C_*$ is a family of pairs $(C_n, d_n)_{n\in\mathbb{Z}}$ where $(C_n)_{n\in\mathbb{Z}}$ is a graded module and $(d_n)_{n\in\mathbb{Z}}$ is a differential on $(C_n)_{n\in\mathbb{Z}}$.

The module $C_n$ is called the module of *n-chains*. The image $B_n = im\ d_{n+1} \subseteq C_n$ is the (sub)module of $n$-boundaries. The kernel $Z_n = ker\ d_n \subseteq C_n$ is the (sub)module of *n-cycles*.

## 2.5 Homology groups

Given a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$, the identities $d_{n-1} \circ d_n = 0$ are equivalent to the inclusion relations $B_n \subseteq Z_n$: every boundary is a cycle but the converse is not generally true. Thus, the next definition makes sense.

**Definition 12.** Let $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ be a chain complex of $R$-modules. For each degree $n \in \mathbb{Z}$, the *n-homology module* of $C_*$ is defined as the quotient module

$$H_n(C_*) = \frac{Z_n}{B_n}$$

There are two different ways of computing homology groups depending on the type of the object. On the one hand, the task of calculating homology groups of a *finite chain complex* (*finitely* many generators at each degree) is translated to a problem of diagonalizing certain matrices called *incidence matrices*, see [22]. On the other hand, in the case of *non-finite type* chain complexes, Sergeraert's effective homology [20] theory provides a framework where this question can be handled. Roughly speaking, the effective homology method links a non-finite type object, $X$, with a finite type object, $Y$, with the same homology groups; then the problem of computing the homology groups of $X$ is reduced to the task of diagonalizing the *incidence matrices* of $Y$.

In this report, we are going to focus on the case of finite objects since the objects coming from digital images are always of finite type.

Once, we have presented all the notions, we can introduce the links between them.

### 2.6 From Simplicial Sets to Chain Complexes

**Definition 13.** Let $K$ be a simplicial set, we define the *chain complex associated with* $K$, $C_*(K) = (C_n(K), d_n)_{n\in\mathbb{N}}$, in the following way:

- $C_n(K) = \mathbb{Z}[K^n]$ is the free $\mathbb{Z}$-module generated by $K^n$. Therefore an $n$-chain $c \in C_n(K)$ is a combination $c = \sum_{i=1}^{m} \lambda_i x_i$ with $\lambda_i \in \mathbb{Z}$ and $x_i \in K^n$ for $1 \leq i \leq m$;
- the differential map $d_n : C_n(K) \to C_{n-1}(K)$ is given by

$$d_n(x) = \sum_{i=0}^{n} (-1)^i \partial_i(x) \text{ for } x \in K^n$$

  and it is extended by linearity to the combinations $c = \sum_{i=1}^{m} \lambda_i x_i \in C_n(K)$.

**Definition 14.** Given a simplicial set $K$, the *$n$-homology group* of $K$, $H_n(K)$, is the $n$-homology group of the chain complex $C_*(K)$:

$$H_n(K) = H_n(C_*(K))$$

### 2.7 From Simplicial Complexes to Simplicial Sets

**Definition 15.** Let $\mathcal{SC}$ be an (ordered abstract) simplicial complex over $V$. Then the *simplicial set $K(\mathcal{SC})$ canonically associated* with $\mathcal{SC}$ is defined as follows. The set $K^n(\mathcal{SC})$ is $S_n(\mathcal{SC})$, that is, the set made of the simplexes of cardinality $n+1$ of $\mathcal{SC}$. In addition, let $(v_0, \ldots, v_q)$ be a $q$-simplex, then the *face* and *degeneracy* operators of the simplicial set $K(\mathcal{SC})$ are defined as follows:

$$\begin{aligned} \partial_i^q((v_0, \ldots, v_i, \ldots, v_q)) &= (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_q), \\ \eta_i^q((v_0, \ldots, v_i, \ldots, v_q)) &= (v_0, \ldots, v_i, v_i, \ldots, v_q). \end{aligned}$$

That is, the face operator $\partial_i^q$ removes the vertex in the position $i$ of a $q$-simplex, and the degeneracy operator $\eta_i^q$ duplicates the vertex in the position $i$ of a $q$-simplex.

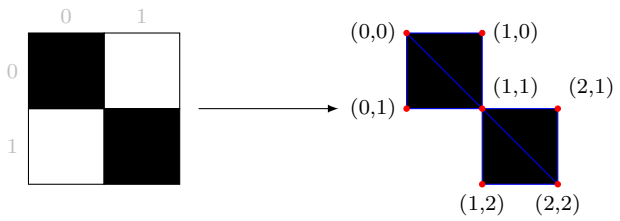


Figure 5: On the left, a digital image; on the right, its simplicial complex representation

**Definition 16.** Given a simplicial complex $\mathcal{SC}$, the *$n$-homology group* of $\mathcal{SC}$, $H_n(\mathcal{SC})$, is the $n$-homology group of the simplicial set $K(\mathcal{SC})$:

$$H_n(\mathcal{SC}) = H_n(K(\mathcal{SC})).$$

## 2.8 From Digital Images to Simplicial Complexes

Let $\mathcal{D}$ be a 2D-image codified as a 2-dimensional array of 1's and 0's. We want to associate a simplicial complex with $\mathcal{D}$. From a digital image, there are several ways of constructing a simplicial complex (see [1]). The approach that we have followed here consists of obtaining from $\mathcal{D}$ the facets of one of its associated simplicial complexes. Subsequently, applying Definition 6 we obtain a simplicial complex associated with $\mathcal{D}$.

The process that we have followed to obtain the facets from a 2D-image $\mathcal{D}$ is as follows. Let $V = (\mathbb{N}, \mathbb{N})$ be the vertex set, that is, a vertex, in this case, is a pair of natural numbers. Let $p = (a, b)$ be the coordinates of a pixel in $\mathcal{D}$ (that is, the position of the pixel in the 2-dimensional array associated with $\mathcal{D}$). From $p$ we can obtain two 2-simplexes that are two facets of the simplicial complex associated with $\mathcal{D}$. Namely, from $p = (a, b)$ we obtain the following facets: the triangles $((a, b), (a+1, b), (a+1, b+1))$ and $((a, b), (a, b+1), (a+1, b+1))$. If we repeat the process for the coordinates of all the pixels in $\mathcal{D}$, we obtain the facets of a simplicial complex associated with $\mathcal{D}$, that will be denoted by $\mathcal{K}_{2D}(\mathcal{D})$.

**Example 17.** Consider the 2D-image depicted in the left side of Figure 5. This image can be encoded by means of the 2-dimensional array: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then, the coordinates of the black pixels are $(0, 0)$ and $(1, 1)$. Therefore, applying our construction we obtain the facets of $\mathcal{K}_{2D}(\mathcal{D})$:

$$(((0, 0), (0, 1), (1, 1)), ((0, 0), (1, 0), (1, 1)), ((1, 1), (1, 2), (2, 2)), ((1, 1), (2, 1), (2, 2))).$$

Once we have the simplicial complex associated with the digital image, we can compute the homology groups of the image from the simplicial complex. As we said previously, several simplicial complexes can be associated with a digital image, but all of them are homomorphic (see [1]); then, we can define the homology groups of a 2D-image as follows:

**Definition 18.** Given a 2D-image $\mathcal{D}$, the *$n$-homology group* of $\mathcal{D}$, $H_n(\mathcal{D})$ is the $n$-homology group of the simplicial complex $\mathcal{K}_{2D}(\mathcal{D})$:

$$H_n(\mathcal{D}) = H_n(\mathcal{K}_{2D}(\mathcal{D})).$$

# 3 Formalizing Algebraic Topology

## 3.1 Digital Images

2D binary monochromatic images were already codified in Coq as binary matrices in the work [19].

In our Coq/SSReflect formalization, we have defined a digital image as a sequence of sequences of boolean elements.

In our ACL2 formalization, we have defined a digital image as a list of lists of 0's and 1's.

A complete description of these formalizations can be seen in [18]

## 3.2 Simplicial Complexes

The notions related to simplicial complexes presented in Subsection 2.2 have been formalized in both the ACL2 Theorem Prover and the proof assistant Coq/SSReflect. A detailed description of these formalizations can be found at [3].

## 3.3 Simplicial Sets

Simplicial sets have been formalized in the ACL2 Theorem Prover. A detailed description of this formalization is described in [10].

## 3.4 Chain Complexes

Chain Complexes have been formalized in the ACL2 Theorem Prover. A detailed description of this formalization can be seen at [15].

## 3.5 Homology groups

Some first step to formalize the computation of homology groups of finite type objects have been given in the proof assistant Coq/SSReflect. A detailed description of this formalization can be found in [11].

## 3.6 From Digital Images to Simplicial Complexes

The construction of the simplicial complex associated with a digital image presented in Subsection 2.8 has been formalized in both the ACL2 Theorem Prover and the proof assistant Coq/SSReflect. A detailed description of these formalizations is presented in [18].

## 3.7 From Simplicial Complexes to Simplicial Sets

Definition 15 was formalized in the ACL2 Theorem Prover. A detailed description of this formalization is provided in [9].

### 3.8 From Simplicial Sets to Chain Complexes

Definition 13 was formalized in the ACL2 Theorem Prover. A detailed description of this formalization can be found in [15].

## 4 Conclusions

In this report we have described the contents of some formalized libraries (in Coq and ACL2) to deal with simplicial complexes and other Algebraic Topology structures. The purpose of these libraries is to be a convenient infrastructure to develop further certified programs to process digital images.

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