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Executive summary

This report details the design and implementation of the dynamic pricing decision support system (DSS) as part of the Integrated Support System for Efficient Water Usage and Resource Management (ISS-EWATUS) project. The software can be found at http://www.vu-bads.org.

The adaptive pricing module is based on models for which the input is based on consumer behaviour data. The objective of this task is to generate predictions on water consumption in terms of changing the water tariffs (pricing schemes) and to compare that with baseline scenario. The adaptive pricing module is meant for strategic level decision makers to assess the impact of different pricing schemes. The DSS is able to assess:

- possible drop of water consumption as a result of changing the water tariffs,
- social affordability of predicted tariffs,
- financial effect of proposed changes for water providers,
- consistency and compatibility of proposed changes with EU law (users pays principle).

The report describes how these goals are achieved through the design of the system in terms of its data models and its deployment model. Furthermore, the architectural, logical and physical overview provides details on the implementation. We provide a user manual for the software as well as descriptions on how to interpret the results. We conclude the document with a validation plan for the DSS.

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1. Introduction

This report details the design and implementation of the dynamic pricing decision support system (DSS) as part of the Integrated Support System for Efficient Water Usage and Resource Management (ISS-EWATUS) project. The software can be found at http://www.vu-bads.org. It is part of work package 6 (WP6) of the project. The goal of this work package is to develop and simulate adaptive water price systems.

2. The scope of the system

The adaptive pricing module is based on models for which the input is based on consumer behaviour data. The objective of this task is to generate predictions on water consumption in terms of changing the water tariffs (pricing schemes) and to compare that with baseline scenario. The adaptive pricing module is meant for strategic level decision makers to assess the impact of different pricing schemes. The main objectives are achieved through the following tasks.

2.1. Main objectives and relations to the other modules of ISS-EWATUS

Task 6.1 Analysis of different datasets to understand water consumption data (M1-M12)

The understanding of residential water consumption data is key to building adaptive pricing policies. There are many datasets that need to be combined in order to identify the key factors in water demand, e.g., analysis of household and municipal data (prices, water usage), socio-demographic data (income, population, composition, household size), climatic data (rainfall, temperature). Based on data mining and descriptive statistics on datasets from Poland, Greece or other one can derive the average consumption per municipality based as a function of different factors, which in turn can be related to price elasticity.

Task 6.2 Design of adaptive pricing models (M6-M18)

The development of adaptive pricing models will use the output of Task 6.1 as an input. The models will incorporate both a resource constraint as well as a budget constraint, which may have consequences on the optimal pricing scheme. This is because when there is a severe water shortage, the resource constraint dominates the budget constraint, so the budget constraint may be irrelevant for the solution. Indeed, optimal prices may be set such that the more elastic demand ends up with a higher price. Incorporating a resource constraint depends on the water-balance equation, since the stock of water in a reservoir net of water saved for next period is to be released in the current period. In addition to the constraints, different models will be developed with alternative pricing structures (block tariffs, flat rates, and variable rates). The models that are developed will be stochastic in nature.

Task 6.3 Development of simulation program (M10-M24)

The adaptive pricing models result in high-dimensional models that are not analytically tractable. Therefore different means are needed for assessing the impact of alternative pricing mechanisms. In this task, a simulation model will be developed to assess the pricing mechanisms and to perform a sensitivity analysis. The results will be used to gain insights into which of the factors (among those identified in Task 6.1) have the greatest influence on water consumption. The simulation program also enables comparison of the economic efficiency of information usage in the pricing models both on a macro-economic as well as micro-economic level (e.g., demand policing).

Task 6.4 Application of the DSS for policy making (M24-M36)

This task is mainly concerned with applying the DSS based on the input data and the adaptive pricing models. This task will also generate the pricing schemes for different scenarios. The DSS aims to provide policy makers and governments, which supply water to households, an effective tool to assess different outcomes for the near future under alternative pricing schemes.

2.2. Requirements addressed by the system

The DSS should be able to assess:

- possible drop of water consumption as a result of changing the water tariffs,
- social affordability of predicted tariffs,
- financial effect of proposed changes for water providers,
- consistency and compatibility of proposed changes with EU law (users pays principle).

3. Design of the system

In this section we explain the design of the system in terms of three aspects of the DSS. We start with the data model that is underlying the DSS. This is the essential data on which the DSS is based. Then we proceed to explain the dynamic nature of the DSS. Finally, we describe how these elements lead to the deployment infrastructure of the DSS.

3.1. Data model

The data model of the DSS consists of three tables on which the output of the DSS is based on. These three tables are depicted in Figure 1.

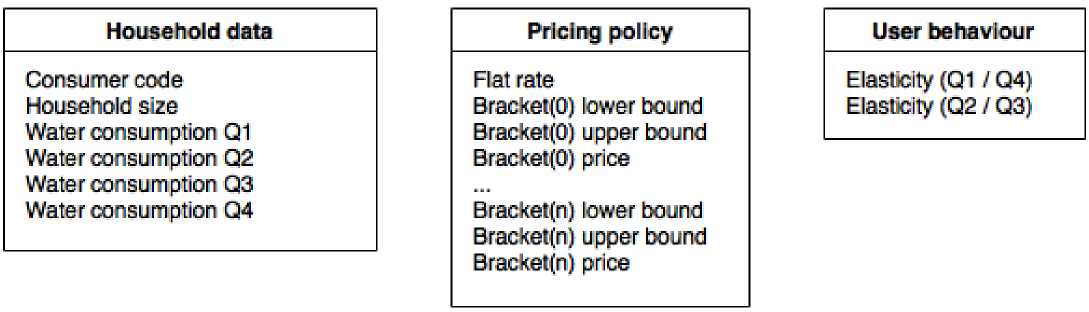


Figure 1: Data model of the DSS

The table with household data contains water consumption data on a specific population in a specific period containing at the least four different quarters of the year. It has information on the household, identified by the consumer code, such as the household size (i.e., number of people in the household) and the total water consumption in m^3 for the four quarters.

The table with data on the pricing policy contains information on the pricing scheme that was applied to the households in the household data table. The pricing scheme consists of the flat rate and several brackets that describe which price was charged for the part of the water consumption that fell within the lower bound and the upper bound of the bracket. These parameters are typical for a progressive pricing scheme. When this data is combined with the data in the table household data, one can calculate the water bill of each household in each quarter.

The table with data on the user behaviour contains information on price demand elasticity. This is split into two data fields, the elasticity for quarters 1 and 4 (the normal seasons) and the elasticity for quarters 2 and 3 (the tourist seasons). This distinction is of importance when calculating the effect of a different pricing scheme, since it might be that water consumption behaviour of tourists is different than of regular household in normal seasons. This data is used to calculate the water consumption of the households in the table household data for different pricing schemes. The discussion of the suggested value of the demand price elasticity index is presented in Appendix 2.

Next to the three tables, the DSS maintains a table for session control. This is necessary to retain input values of clients over different web pages. The session control table stores an incremental ID, a session ID, data for the session, and the last date of the active session. This is standard session control as is commonly used in web applications.

The data described in the data model are stored in an online database. These data can be changed and updated according to new insights and information. The dynamic pricing module reads in this data every time before a webpage is built. Hence, it assembles the final page that is shown to the user dynamically based on the latest data. Also, the input from the user, i.e., a different price elasticity or a different pricing scheme, is stored as a session to retain the input values of different pages. When first connecting to the module, the application checks is the client corresponds to a previously stored session. If this is the case, then the stored input values are used instead of the standard values in the database.

3.2. Deployment model

The deployment model of the DSS is given in the following picture, Figure 2. The setup of the dynamic pricing module is that a client connects to the dynamic pricing module through the internet over the TCP/IP protocol. The request is handled by an HTTP daemon (in our case, an Apache web server). The module page is then dynamically created after reading relevant data from a MySQL database. The functionality after creation of the first page is further done through PHP on the server side and Javascript on the client side.

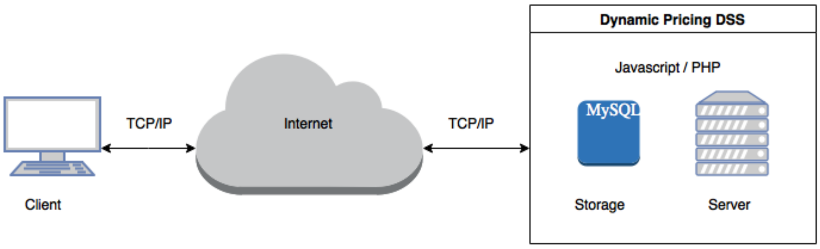


Figure 2: Deployment model of the DSS.

4. Architecture of the system

In this section we describe the architecture of the DSS. We first provide an overview of the submodules of the DSS and describe their functions. Then we provide a logical architecture overview for each of the modules. Finally, we end with a description of the physical architecture overview.

4.1. Architecture overview

The logic of the DSS is separated into four parts of the application logic. These parts are depicted in Figure 3.

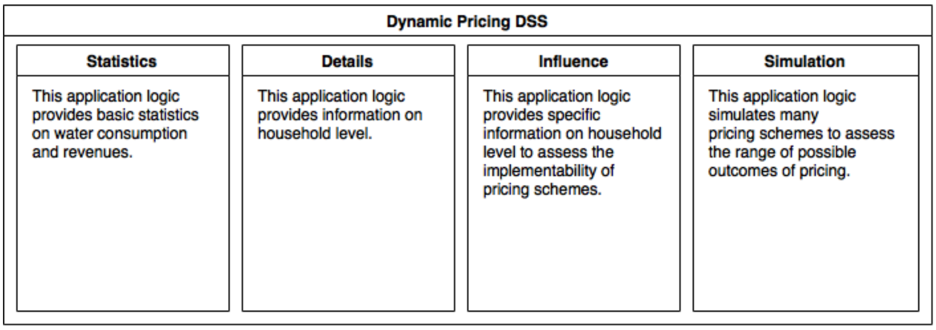


Figure 3: Architecture of the DSS.

The application logic statistics calculates the water consumption and water bill per quarter for each of the households mentioned in the household database. A similar computation is done for the pricing scheme given by the user. These values and the differences therein is given in the statistics module.

The application logic details breaks down the statistics of the application logic statistics into detailed information per household. This allows for assess the impact on individual households. The application logic influence is concerned with the implementability of pricing schemes. It creates statistics on how households are affected by new pricing schemes. Which are the households most affected by price increases and which are the ones most influenced in their water consumption.

The application logic simulation generates a lot of pricing schemes and evaluates these schemes. It provides information on the possible performance of pricing. It shows the non-linear relation between revenues and water consumption as a function of the input variables.

4.2. Logical architecture overview – a list of functional modules

The logical architecture of the DSS consists of three layers, see Figure 4. The first layer is the presentation layer, the second is the application logic layer, and the third is the data source layer.

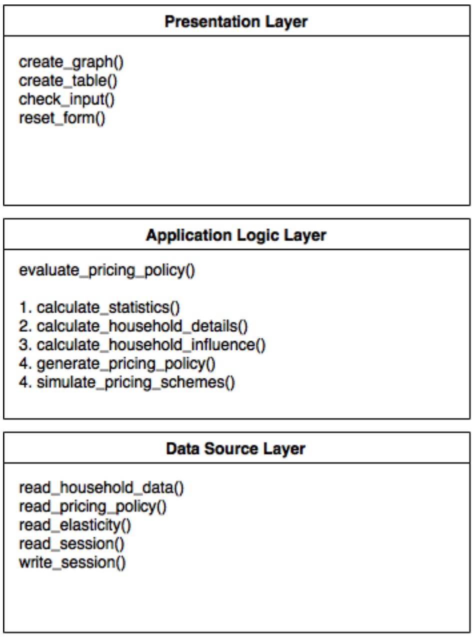


Figure 4: Logical architecture overview.

The presentation layer contains functions that builds elements on the web page to show the client. The functions that are implemented in this layer pertain the creation of graphs, tables, and checking and passing on information from user forms. The rest of the presentation is taken care of by static elements of the web page (e.g., HTML and CSS).

The application logic layer implements a function for the evaluation of a pricing policy. However, other functions differ in the implementation for each of the sub modules described in the previous subsection. In the first module (statistics), one main function for the calculation of the statistics is implemented. This is similar for the second module (details) and the third module (influence). The last module (simulation) has a function for the generation of a pricing policy that satisfies certain constraints and for the simulation of pricing schemes.

The data source layer implements functions that deal with reading data from the data model. Moreover, it has implementation of function for reading and writing session data.

4.3. Physical architecture overview

The physical architecture can be seen in Figure 2. The DSS is a web-based system that interacts with the client. The whole DSS runs on a single server that runs an Apache web server, a MySQL database, and is PHP-enabled.

5. Implementation details

The structure of pricing and the effect on revenues and water consumption is very intricate and difficult to assess analytically. The step to optimization adds even more to this complexity. We have adopted theory from ordinal optimization to approach this complexity and to quantify the complexity of water pricing.

The essential idea of ordinal optimization is simulate pricing policies uniformly from the space of all pricing policies. Let a pricing policy be denoted by θ from the space Θ of all pricing policies. Let the economic indicators of the policy θ be denoted by $J(\theta)$. The optimization problem can be stated as $\max_{\theta\in\Theta} J(\theta)$ which turns out to be $\mathrm{E}\, L(\theta,\xi)$, the expectation of the sample performance L as a function of θ and the randomness in the system ξ. The problem is that the search space Θ is a huge and perhaps with little structure limiting the performance of search-based methods.

Two essential ideas in our approach lead to a feasible technique to handle the complexity. First, the order of two alternative pricing policies (i.e., which one is better) converges exponentially fast while the value (i.e., the exact difference in performance) converges at rate $1/\sqrt{N}$, where N is the number of simulations. The second observation is that goal softening eases the computional burden of finding the optimum. In this case, one does not go for the optimal solution in Θ, which is a single alternative in the set. Instead, the objective is to find a pricing policy that is in the top-n% of the choice for 95% of the time. This probability improves dramatically with the sizes of these sets.

The above method allows one to come up with a Monte Carlo method to optimize pricing policies. At the same time one can quantify the complexity of pricing in practice through ordered performance curves (OPC curves). These are plots of the values of J as a function of the order of performance (i.e., the best, the second best, and so on). The OPC curve is robust against noise and displays the order of various complexities. In the following table we list the main functions in the software to achieve these goals.

Function: read_household_data()
Input: database, table, login, password *Output*: data structure with records on household, number of occupants, water consumption per quarter *Description*: this function reads stored data from a specified table on households containing records on a household specified by an ID, the number of occupants in the household, the water consumption per quarter of this household.

Function: read_pricing_policy()

Input: database, table, login, password

Output: data structure with records on the pricing policy used for the baseline scenario

Description: this function reads stored data from a specified table on the pricing policy used in the baseline scenario of the pricing module. The pricing policy consists of the flat rate, the lower and upper brackets for the volumetric usage with its corresponding price within that bracket.

Function: read_elasticity()

Input: database, table, login, password

Output: data structure with values of the price demand elasticity for the baseline scenario

Description: this function reads stored data from a specified table on the price demand elasticity used in the baseline scenario of the pricing module. The elasticity values contain values for the normal season and the touristic season.

Function: read_session()

Input: session_ID

Output: session data

Description: this function reads in session data that corresponds to the user with session_ID. The data is stored in a pre-specified table, specifically meant for session data. The session data consists of values for the price elasticity, the flat rate, the lower and upper brackets with its corresponding price.

Function: write_session()

Input: session_ID, session_data

Output: none

Description: this function writes session data that corresponds to the user with session_ID to a pre-specified table, specifically meant for session data. The session data consists of values for the price elasticity, the flat rate, the lower and upper brackets with its corresponding price.

Function: evaluate_pricing_policy()

Input: baseline_pricing, new_pricing, household_data

Output: new_household_data

Description: this function uses the baseline pricing scheme and the new pricing scheme to generate new household data that can be compared to the old household data. Let the pricing scheme in the baseline scenario be determined by the flatrate fr and the lower brackets $lb(i)$ and upper brackets $ub(i)$ with corresponding price $p(i)$. The expenditure e for a household with water consumption wc will be $e = fr + \sum_{i=1}^{n} \min\big(max(wc - lb(i), 0)\,, ub(i)\big) * p(i)$. A similar calculation can be done with the new pricing scheme with fr', $lb'(i)$, $ub'(i)$, and, $p'(i)$ as input. This gives a new expenditure e' given by the equality $e' = fr' + \sum_{i=1}^{n} \min\big(max(wc - lb'(i), 0)\,, ub'(i)\big) * p'(i)$. The relative difference in the expenditure is then given by $diff = (e' - e)/e$. Based on this difference the new water consumption wc'can be calculated by $wc' = wc * pde$, where pde is the price demand elasticity. When this is done for every household in the household data, then many statistics can be calculated. This includes the total water consumption by the complete population in the household data per quarter, the total revenues generated by that population before and after the change in the pricing scheme.

Function: calculate_statistics()

Input: baseline_pricing, new_pricing, household_data

Output: total_water_consumption, total_revenue_before_change, total_revenue_after_change

Description: this function relies on evaluate_pricing_policy(), which is called first. After the call to this function a table with household data is acquired with respect to the new pricing scheme. Hence it is known what e, wc, e', and wc' are. Based on wc' one can calculate e'', the real expenditure due to effects of the price elasticity, and the relative difference with the baseline scenario. These statistics are reported by aggregating household data over different quarters, and over all quarters.

Function: calculate_household_details()

Input: baseline_pricing, new_pricing, household_data

Output: new_household_data

Description: this function relies on evaluate_pricing_policy(), which is called first. After the call to this function a table with household data is acquired with respect to the new pricing scheme. Hence it is known what e, wc, e', and wc' are. Based on wc' one can calculate e'', the real expenditure due to effects of the price elasticity, and the relative difference with the baseline scenario. These statistics are reported for each household. The data structure used to represent this data has filters attached to them such that data can be easily sorted on the different statistics.

Function: calculate_household_influence()

Input: baseline_pricing, new_pricing, household_data

Output: new_household_data, data structure with the difference in water consumption and expenditure per household

Description: this function relies on evaluate_pricing_policy(), which is called first. After the call to this function a table with household data is acquired with respect to the new pricing scheme. Hence it is known what e, wc, e', and wc' are. Based on wc' one can calculate e'', the real expenditure due to effects of the price elasticity, and the relative difference with the baseline scenario. The statistics on $e'' - e$, $(e'' - e)/e$, $(wc' - wc)/wc$ are reported as key statistics to base the economic implementability on.

Function: generate_pricing_policy()

Input: max_upper_bracket, max_price, number_of_brackets

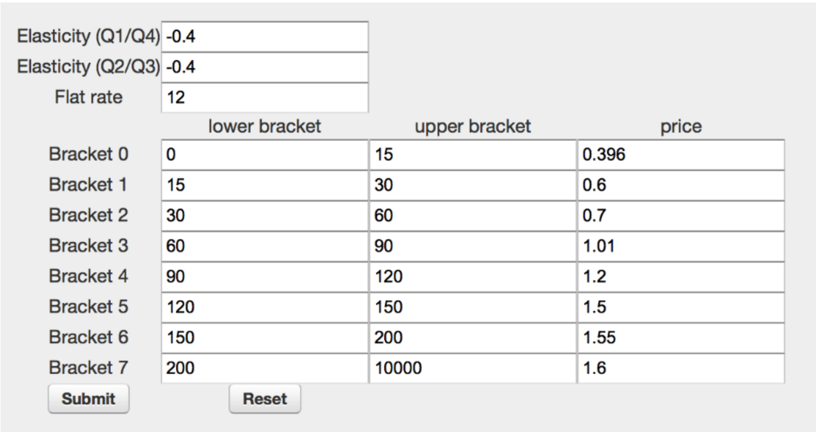
Output: pricing_policy

Description: this function generates a pricing policy uniformly drawn from the space Θ of pricing policies. Let n be the number of brackets that are needed to be generated. Then the i-th upper bracket is generated be drawing from a uniform $U[ub(i-1), max\_price]$ distribution, where $ub(0)$ = 0. The lower brackets are generated by $lb(1) = 0$, and $lb(i) = ub(i-1)$ for $i > 1$.

Function: simulate_pricing_schemes()
Input: number_of_simulations *Output*: data structure with the policies and statistics *Description*: this function relies on the function generate_pricing_policy(), evaluate_pricing_policy(), and create_graph(). Let n be the number of simulation is specified as input for the function. Then, the function generates n times a pricing policy that is uniformly drawn from the space Θ of pricing policies, evaluates it, and stores it in a data structure. This data structure is passed on to create_graph() to plot a Pareto frontier.
Function: create_graph()
Input: data structure with pairs of (x, y) coordinates *Output*: an HTML data structure for graphs *Description*: this function converts a data structure with (x, y) coordinates to a graph to be included on an HTML page as a javascript object. The framework used for this function is the highcharts.js framework.
Function: create_table()
Input: data structure with information represented in tuples *Output*: an HTML data structure for tables *Description*: this function converts a data structure with information represented in tuples to a table to be included on an HTML page as a javascript object. The framework used for this function is the datatable.js framework. The framework applies search capabilities as well as filters to the tables.
Function: check_input()
Input: none *Output*: none *Description*: this function in invoked when pressing a button in the software and it is attached to the behavior of the elements in the user input module. The function guarantees that the user input is valid, i.e., that all data is filled in, $0 = lb(1) < ub(1) = lb(2) < ub(2) = \cdots = lb(n) < ub(n)$, and that $p(1) \leq \cdots \leq p(n)$. In case these constraints are violated and error is thrown and the user input form is not serialized and processed.
Function: reset_form()
Input: none *Output*: none *Description*: this function in invoked when pressing the reset button in the software. It makes sure that the user input form is filled in with values corresponding to the baseline scenario.

6. User manual

The dynamic pricing module consists of four tabs with various statistics on pricing schemes. The main page has a focus on global characteristics of pricing policies. The input to the module consists of the price demand elasticity for a normal season (Q1/Q4) and the tourist season (Q2/Q3) (i.e., how does water consumption by consumers change as the price changes), and the pricing policy (which consists of the flat rate and the prices as a function of different water consumption brackets). The user can input these quantities in the grey area on the main page of the module, see Figure 5 below. The prices are specified in euros per m^3, and the brackets indicate the region in m^3/calculation period (quarter) in which the price is in effect.



Elasticity (Q1/Q4)	-0.4		
Elasticity (Q2/Q3)	-0.4		
Flat rate	12		
	lower bracket	upper bracket	price
Bracket 0	0	15	0.396
Bracket 1	15	30	0.6
Bracket 2	30	60	0.7
Bracket 3	60	90	1.01
Bracket 4	90	120	1.2
Bracket 5	120	150	1.5
Bracket 6	150	200	1.55
Bracket 7	200	10000	1.6
Submit	Reset		

Figure 5: Input parameters of the module.

Once the parameters have been given by the user, the parameters can be submitted to the module by the submit button. The tool will then display the new pricing policy together with the benchmark policy. The benchmark policy is the policy that was in use a specific year to create a benchmark with the new pricing policy. One can see an example of this graph in Figure 6.

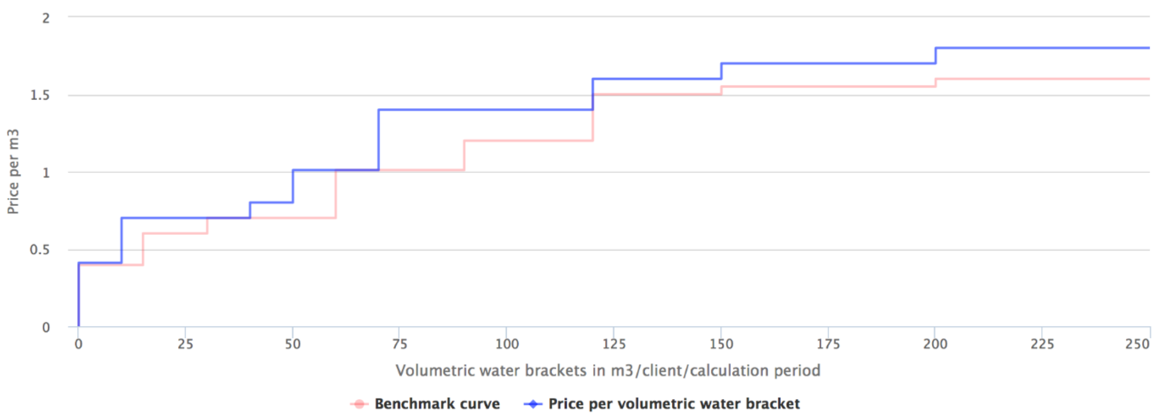


Figure 6: Example of pricing scheme.

Statistics of the new pricing policy are calculated automatically and are depicted in a table on the main page. The table includes the total water consumption in m^3 over a year, as well as the breakdown into the different quarters of the year. It also lists the total revenue based on the water consumption not considering any effect of demand elasticity of consumers. This gives an intuitive idea of how much one would receive in revenues by changing the pricing schemes while everything else remains constant (this quantity is also specified per quarter). However, a change in the pricing schemes comes with changes in water consumption. Hence, the total revenue that one really collects is different. This is listed in the total revenue after change in the water consumption, which has also a breakdown in quarters.

In the following example (see Figure 7) one can see that a new pricing scheme is in effect. The scheme is such that if the water consumption would remain the same (at the level of 51,949 m^3 per year) for this population, then the total revenue would increase by 11.46% (from 68,216.10 euro to 76,034.28 euro). However, due to the fact that the water consumption decreases with 9.22% the total increase in revenues is actually less than 11.46%. It turns out that the revenues increase only by 0.27%. This example directly illustrates the difficulty in assessing pricing schemes. In cases where one expects that the revenues go up while at the same time water usage is reduced, it turns out that in practice different numbers correspond to reality. This warrants the development of a dynamic pricing tool.

Statistic	Benchmark	Current	Rel. difference (in %)
Total water consumption (in m3)	51949.00	47158.20	-9.22
Total water consumption Q1 (in m3)	10543.00	9474.56	-10.13
Total water consumption Q2 (in m3)	13855.00	12551.09	-9.41
Total water consumption Q3 (in m3)	17395.00	15939.41	-8.37
Total water consumption Q4 (in m3)	10156.00	9193.14	-9.48
Total revenue before change in water consumption (in euros)	68216.10	76034.28	11.46
Total revenue Q1 (in euros)	13113.26	14418.27	9.95
Total revenue Q2 (in euros)	17175.33	19289.76	12.31
Total revenue Q3 (in euros)	25052.34	28387.21	13.31
Total revenue Q4 (in euros)	12875.16	13939.04	8.26
Total revenue after change in water consumption (in euros)	68216.10	68402.02	0.27
Total revenue Q1 (in euros)	13113.26	13045.30	-0.52
Total revenue Q2 (in euros)	17175.33	17101.11	-0.43
Total revenue Q3 (in euros)	25052.34	25447.62	1.58
Total revenue Q4 (in euros)	12875.16	12807.98	-0.52

Figure 7: Statistics of a new pricing scheme.

The tab with the details gives insight in the underlying data that the dynamic pricing tool works with. It is based on consumer data on water usage over a specific period. The table shows the customer code, the number of people in the household of the consumer, the water usage over the different quarters, the water bill in each quarter, the new price for the consumer of the new policy per quarter, and the water usage based on the new price per quarter. The columns BM Cons 1 to BM Cons 4 denote the water consumption in m^3/household for that quarter in the baseline scenario. The columns BM WB Q1 to BM WB Q4 depict the expenditures in the baseline scenario in euros for that quarter. The columns WB Q1 to WB Q4 are the expenditures in the new scenario in euros for that quarter. Finally, Cons Q1 to Cons Q4 denote the water consumption in the new scenario in m^3 per household. The data on this page essentially aggregates to the statistics on the main tab. The dynamic pricing tool internally calculates for each household the effect of price changes using the price demand elasticity and then aggregates this data on the statistics tab. Figure 8 gives an impression of part of the data on the detailed level.

Customer Code	Household	BM Cons Q1	BM Cons Q2	BM Cons Q3	BM Cons Q4	BM WB Q1	BM WB Q2	BM WB Q3	BM WB Q4	WB Q1	WB Q2	WB Q3	WB Q4	Cons Q1	Cons Q2	Cons Q3	Cons Q4
0049300000	3	16.00	11.00	1.00	14.00	20.21	17.83	13.51	19.12	21.32	18.05	13.51	20.01	13.80	10.50	1.00	12.14
0049400000	2	151.00	62.00	44.00	68.00	316.20	65.79	49.20	76.98	314.04	63.63	47.70	74.82	151.27	63.31	45.22	69.12
0054500000	4	23.00	58.00	106.00	77.00	24.79	60.54	156.52	93.75	25.90	58.38	154.37	91.59	21.21	59.43	106.55	77.92
0056500000	4	47.00	58.00	101.00	37.00	52.15	60.54	144.48	39.28	49.99	58.38	142.32	40.40	48.66	59.43	101.60	35.87
0057000000	3	24.00	54.00	161.00	27.00	25.44	57.49	362.52	27.40	26.55	55.33	360.36	28.51	22.25	55.50	161.24	25.38
0057500000	4	1.00	20.00	35.00	2.00	13.51	22.82	36.45	13.94	13.51	23.94	37.56	13.94	1.00	18.05	33.78	2.00
0060590000	2	32.00	62.00	90.00	18.00	32.20	65.79	117.98	21.52	33.31	63.63	115.82	22.63	30.62	63.31	90.73	15.93
0064000000	2	20.00	83.00	124.00	25.00	22.82	104.93	206.42	26.09	23.94	102.78	204.27	27.21	18.05	83.82	124.42	23.30
0064100000	5	21.00	32.00	32.00	25.00	23.48	32.20	32.20	26.09	24.59	33.31	33.31	27.21	19.11	30.62	30.62	23.30
0069500000	3	25.00	36.00	41.00	30.00	26.09	37.87	44.95	29.36	27.21	38.98	45.41	30.48	23.30	34.83	40.59	28.49

Figure 8: Detailed information on the household level.

When a pricing scheme has been set the main page calculates several statistics, of which the most important ones are the effect on water consumption and total revenues. However, for a complete picture, it is necessary to also evaluate how this impacts the household on an individual level. The details tab already give some insight into individual behaviour, however, the influence tab allows for further analysis. The table on this tab provides a sorted overview of the top 10 households that are affected most in several ways. One can select the top 10 households that have the highest expenditure, but also the top 10 households that have the highest reduction in water consumption. The table lists the costumer code, the number of people in its household, the total water consumption and water bill under the benchmark policy, and the total water consumption and water bill under the new pricing policy. The last two columns display the difference in expenditure of the consumer (in %) and the difference in water consumption (in %). Both of these columns can be sorted on to generate different top 10 listings.

Figure 9 depicts are screenshot of the table. One can see that the household set by this policy are affected such that there is an increase in expenditures of at most 5.59%. At the same time, this household, as a result of price changes, will use 14.44% less water.

Customer Code	Household	BM Consumption	BM Waterbill	Consumption	Waterbill	Rel. exp.diff. (in %)	Rel. cons.diff. (in %)
1337000000	2	62.00	79,53	53.05	83.97	5.59	-14.44
1288000000	2	65.00	81,49	56.25	85.94	5.46	-13.46
1103500000	2	69.00	84,10	60.53	88.55	5.29	-12.27
1116500000	2	72.00	86,07	63.70	90.51	5.17	-11.52
1356500000	3	75.00	88,03	66.81	92.48	5.05	-10.91
1090310000	2	75.00	88,03	66.81	92.48	5.05	-10.91
1166500000	2	75.00	88,03	66.88	92.48	5.05	-10.83
1120000000	1	77.00	89,34	68.93	93.78	4.98	-10.48
1104000000	2	64.00	81,28	56.17	85.28	4.92	-12.23
1365500000	1	58.00	77,58	50.24	81.36	4.87	-13.38

Figure 9: Influences on the household level.

There are many different pricing schemes that can be devised and evaluated. The different combinations of the flat rate, the volumetric brackets and the respective prices therein are immense. Therefore, one needs to be assisted in the evaluation of different pricing schemes. The simulation tab provided help in this requirement. This module simulates a large number of randomly generated pricing policies for a given price demand elasticity and flat rate. It change the volumetric brackets and the prices therein. For each simulation the module records several statistics, in particular, the difference in water consumption (in %) and the difference in revenues (in %) simulated over a period of one year. The module displays the Pareto frontier of the different pricing schemes, i.e., all combinations of the two performance indicators.

Figure 10 displays the Pareto frontier for a specific setting (in this case, an elasticity of -0.4 and a flat rate of 12 euro). The results show that it is hard to obtain both an increase in revenues and reduction in water consumption at the same time. One the one hand, an increase in prices has such an impact on the reduction of water consumption that it will not generate additional revenues. On the other hand, lowering pricing is also a possibility. This will increase the water consumption, but will not generate sufficient demand that additional revenues are generated. Hence, the curvature of the graph.

Note that there a few policies that do attain a higher revenue while at the same time also reduce water consumption (there is a point at -6.5% in water consumption and 1.09% in revenues), however, the final result depends on the value of demand price elasticity index. These are quite rare though and indicate that setting a good pricing schemes is a difficult problem that needs to be approached with considerable care.

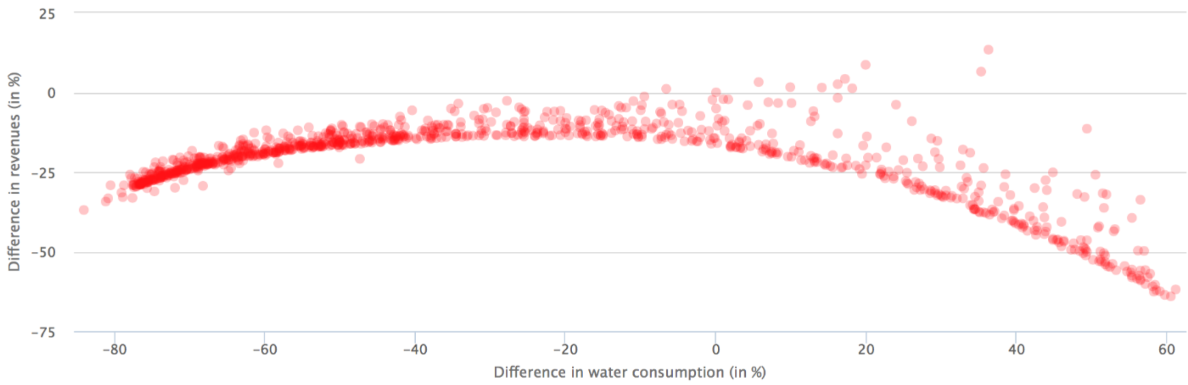


Figure 10: Pareto frontier of different pricing schemes.

The price demand elasticity is a very important factor in this analysis. If the elasticity changes from the value -0.4 to -0.3, then there are more policies that attain better performance in water consumption and revenues simultaneously. In Figure 11 we can see how the graph changes as the elasticity changes. It clearly shows that there are more points above the zero-line for the revenues. A similar analysis can be done with the flat rate.

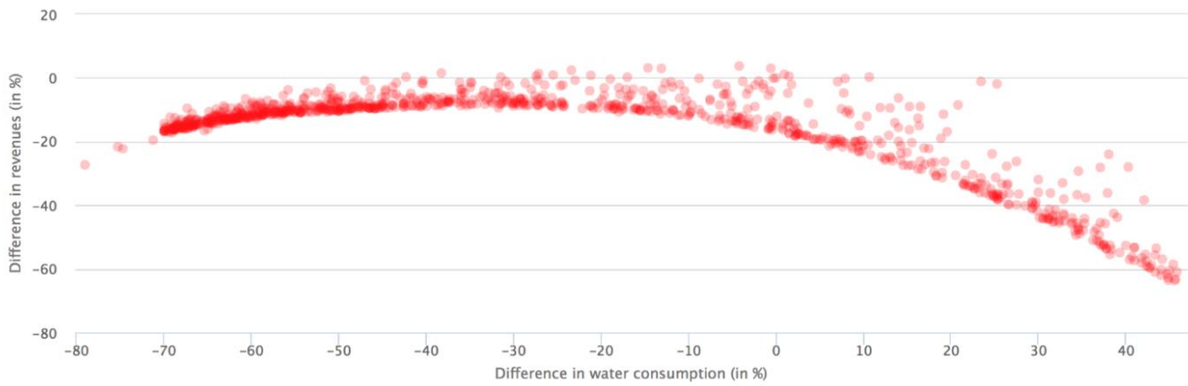


Figure 11: Pareto frontier of different pricing schemes with reduced price elasticity.

The graph depicts the realm of possibilities of different pricing schemes. In order to get more insights in the type of policies that obtain these result, a table is presented with all simulated policies and there performance. The table lists the policy, identified by a vector of the brackets and the price therein, the difference in water consumption (in %), the difference in revenues when no difference in consumption is taken into account (in %), and the expected revenues taking into account the changes in water consumption (in %).

Figure 12 shows a screenshot of the table. There are filters for sorting the results on the different performance indicators. Currently, the table is sorted on the final revenues. The discrepancy between the different revenue values shows that the changing a pricing scheme while ignoring the elasticity in water consumption has potential threats. The ninth policy shows that in the pricing scheme one would expect an increase in revenues if one ignores consumer behavior, however, in practice one would observed a decrease in revenues. Note that the fourth policy achieves both a reduction in water consumption while at the same time the revenues are increased.

Policy (brackets, prices)	Diff. water.cons.	Diff. revenues	Diff. final revenues
(0,35,130,262,292,294,296,297), (0.075,1.812,1.946,1.989,1.992,1.997,2,2)	32.10	-17.72	7.81
(0,36,43,95,198,280,291,298), (0.197,1.501,1.835,1.852,1.971,1.974,1.996,1.998)	12.97	6.20	4.04
(0,30,45,117,148,248,277,294), (0.298,0.651,1.552,1.867,1.91,1.935,1.999,2)	14.61	-6.34	3.07
(0,48,105,146,175,190,213,248), (0.632,1.488,1.622,1.72,1.737,1.814,1.869,1.872)	-9.30	6.25	1.35
(0,57,91,130,295,296,296,299), (0.579,1.741,1.997,1.997,2,2,2,2)	-3.41	3.29	0.88
(0,19,229,256,290,300,300,300), (0.011,1.622,1.656,1.729,1.912,1.985,1.991,1.992)	2.73	4.33	0.86
(0,32,83,158,247,282,286,295), (0.433,1.983,1.985,2,2,2,2,2)	-7.63	16.51	0.77
(0,45,103,117,181,288,299,299), (0.477,1.226,1.727,1.847,1.961,1.986,1.996,1.997)	6.11	-5.89	-0.10
(0,32,36,107,209,246,262,271), (0.389,0.673,1.783,1.925,1.952,1.979,1.989,1.996)	-2.88	11.71	-0.12
(0,27,162,167,295,298,300,300), (0.389,1.584,1.982,1.984,1.99,1.996,1.996,1.999)	-3.70	5.97	-0.40

Figure 12: Details on the simulated policies.

The dynamic pricing tools allows to simulate different pricing schemes under different assumption. It shows that the dynamic pricing problem is a hard problem in which a lot of care needs to be taken. The policies that are simulated need to be judged together with the information on the influences tab. One need to find a balance between water reduction, increase in revenues, and feasibility of the policy in terms of influences and fairness. The dynamic pricing tool assists in finding this balance and has the potential to be integrated in important pricing policy decisions at a strategic level.

7. Technical requirements and development environment

The DSS does not require any specific technical requirements on the server than the ones described in the architecture above. A regular server with common specifications connected to the internet will suffice to run the DSS.

8. Testing

In order to test the module, one needs to address the influence on KPIs and the pricing schemes. The most important KPIs are: changes of water consumption due to saving water and the related financial effects (change in total revenues, change in expenditure for purchasing water services).

Taking in to account the primary target of the project – reduction of water consumption and in this way balancing the demand with available resources – the methodology for testing will be focused on the assessment of the water savings using the economic instrument (water tariffs). The general approach can be described as dQ = f(dP), where dQ are the changes of the quantity of a raw good or services (water supply in this case), and dP is the change of the price for good/service.

Of course there are many other factors influencing on changes in quantity of consumed water but this module is focused on economic instruments. In more detail, but still focusing only on the economic aspects, the following formula describes the responsiveness of the quantity of a raw good or service demanded to changes in its price $dQ = e_p * dp / P * Q$, where e_p is the coefficient of price elasticity of demand, P is the initial price, Q is the quantity demanded before changes of the price in the calculation period.

In case of water demand, the term "price" is more complicated especially for a mixed tariff that consists of a flat rate and volumetric charge. Therefore, the real price of water is derived using formula $P = E_x / Q$, where E_x is the total expenditure per client in the calculation period (E_x = FLr + Vch, where FLr is the flat rate in the calculation period and Vch is the volumetric charge (quantity or quantities multiplied by price or prices).

The desk review gives us the estimation of the e_p in local conditions, the other data was derived from water operators. Such methodology creates the possibility of predicting water saving by introducing changes in water tariffs. Such effects are related to an increase of water tariffs, however constructed model allows to review and check the affordability aspect.

Appendix 1: survey on dynamic pricing

1. Introduction

Dynamic pricing is the study of determining optimal selling prices of products or services, in a setting where prices can easily and frequently be adjusted. This applies to vendors selling their products via Internet, or to brick-and-mortar stores that make use of digital price tags. In both cases, digital technology has made it possible to continuously adjust prices to changing circumstances, without any costs or efforts. Dynamic pricing techniques are nowadays widely used in various businesses, and in some cases considered to be an indispensable part of pricing policies.

Digital sales environments generally provide firms with an abundance of sales data. This data may contain important insights on consumer behavior, in particular on how consumers respond to different selling prices. Exploiting the knowledge contained in the data and applying this to dynamic pricing policies may provide key competitive advantages, and knowledge how this should be done is of highly practical relevance and theoretical interest. This consideration is a main driver of research on dynamic pricing and learning: the study of optimal dynamic pricing in an uncertain environment where characteristics of consumer behavior can be learned from accumulating sales data.

The literature on dynamic pricing and learning has grown fast in recent years, with contributions from different scientific communities: operations research and management science (OR/MS), marketing, computer science, and economics/econometrics. This survey aims at bringing together the literature from these different fields, and at highlighting some of the older (and sometimes forgotten) literature where many important results and ideas already can be found.

A few literature reviews on dynamic pricing and learning do already exist. Araman and Caldentey [2] and Aviv and Vulcano [3, Section 4] review in detail a number of recent studies, mostly from the OR/MS community; Christ [4, Section 3.2.1] contains an elaborate discussion of a selection of demand learning studies; and Chen and Chen [5] review recent research on multiple-product pricing, pricing with competition, and pricing with limited demand information. Our survey complements these publications by aiming at a larger scope, and, although our main focus is on the OR/MS literature, we also address relevant contributions from computer science, marketing, economics and econometrics.

Content. This survey reviews the literature on dynamic pricing with demand uncertainty. We discuss how this is embedded in the literature on dynamic pricing in general, but do not review all relevant research topics associated with dynamic pricing; for this we refer to Bitran and Caldentey [6], Elmaghraby and Keskinocak [7], Talluri and van Ryzin [8], Phillips [9], Heching and Leung [10], Gönsch et al. [11], Rao [12], Chenavaz et al. [13], Deksnyte and Lydeka [14] and Özer and Phillips [15]. We focus on studies where selling price is a control variable; we only scarcely discuss learning in capacity-based revenue management [8] or learning in newsvendor/inventory control problems. Neither do we consider mechanism design [16, 17] or auction theory with incomplete information (see e.g. [18, 19] and the references therein), although there are some similarities with dynamic pricing and learning.

Most of the studies that we discuss are written from an (online) retailer perspective; we do not consider social welfare optimization [20, 21]. We do not dive into specific details associated with particular application area such as pricing in queueing or telecommunication environments [22], road pricing [23–25], or electricity pricing [26,27], to name a few. We also neglect a recent stream of empirical studies that aims to explain the dynamic pricing strategies of sellers by fitting models to sales data (see e.g. [28] for an example on prices of airline tickets, Sweeting [29] on prices of Major League Baseball tickets, and Huang et al. [30] on pricing for a used-car dealership). Finally, this survey focuses on studies where the seller learns about the demand function, and not on studies where buyers (or sellers) learn (typically about the quality of the product) [31–37].

Methodology. We used Google Scholar to find all relevant references that were available online October 1, 2014. We excluded double versions of the same papers, or conference papers that largely overlap with journal papers. For all papers that we found this way we looked on Google Scholar for relevant other work citing these papers. We also checked the websites of key authors in the field for relevant publications. We aimed for a comprehensive review of the dynamic-pricing-and-learning literature; for the other sections, on e.g. demand estimation or dynamic pricing under full information, we restrict to key papers and reviews.

2. Historical origins of pricing and demand estimation

Dynamic pricing with learning can be regarded as the combined application of two research fields: (1) statistical learning, specifically applied to the problem of estimating demand functions, and (2) price optimization. Both these fields are already quite old, dating back more than a century. In this section we briefly describe the historical fundaments out of which dynamic pricing and learning has emerged, by pointing to some key references on static pricing and estimating demand functions that have been important in the progress of the field.

2.1. Demand functions in pricing problems

Cournot [38] is generally acknowledged as the first to use a mathematical function to describe the price–demand relation of products, and subsequently solve the mathematical problem of determining the optimal selling price. As vividly described by Fisher [39], such an application of mathematical methods to study an economical problem was quite new and controversial at the time, and the work was neglected for several decades. Cournot showed that if F (p) denotes the demand as a function of price p, where F (p) is continuous, decreasing in p, and pF(p) is unimodal, then the price that maximizes the revenue pF(p) can be found by equating the derivative of pF(p) to zero. If F (p) is concave there is a unique solution, which is the optimal price (this is contained in Chapter IV of Cournot [38]). In this way, Cournot was the first to solve a ‘‘static pricing’’ problem by mathematical methods.

2.2. Demand estimation

To make theoretical knowledge on optimal pricing theory applicable in practical problems, one needs to have an estimate of the demand function. The first known empirical work on demand curves is the so-called King–Davenant Law [40] which relates the supply and price of corn (see [41], for an exposition on the origins of this work). More advanced research on estimating demand curves, by means of statistical techniques such as correlation and linear regression, took off in the beginning of the 20th century. Benini [42], Gini [43] and Lehfeldt [44] estimate demand curves for various goods as coffee, tea, salt, and wheat, using various curve-fitting methods. Further progress on methodology was made, among others, by Moore [45,46], Wright [47] and Tinbergen [48]; the monumental work of Schultz [49] gives a thorough overview of the state-of-the-art of demand estimation in his time, accompanied by many examples. Further references and more information on the historical progress of demand estimation can be found in [50, Section II], [51, particularly section iii], [52–54]. A small sample of the many contemporary studies on demand estimation in different contexts is Berry et al. [55], McFadden and Train [56] and Bajari and Benkard [57].

2.3. Practical applicability

Estimating demand curves of various products was in first instance not aimed at profit optimization of commercial firms, but rather used to support macro-economic theories on price, supply, and demand. Application of the developed methods in practical problems was initially far away. An illustrative quote is from [58], who doubted the possibilities of applying the theory of monopoly pricing on practical problems, exactly because of the difficulty of estimating the demand curve:

> It is evidently the opinion of some of the writers under discussion that the modern theory of monopoly is not only capable of throwing considerable light on the general principles underlying an individualistic economic structure, but that it is also capable of extensive use in the analysis of particular practical economic problems, that is to say, in applied economics. Personally, I cannot but feel skeptical about this. [. . .] There does not seem to be any reason why a monopolist should not make a mistake in estimating the slope of the demand curve confronting him, and should maintain a certain output, thinking it was the position which maximized his profit, although he could actually have increased his profit by expanding or contracting. [58, p. 18,19].

Hawkins [59] reviews some of the attempts made by commercial firms to estimate the demand for their products. Most of these attempts were not successful, and suffered from difficulties of obtaining sufficiently many data for reliable estimates, and of changes in the quality of the product and the prices of competitors. Even a very detailed study of General Motors on automobile demand ends, somewhat ironically, with:

> The most important conclusion from these analyses of the elasticity of demand for automobiles with respect to price is that no exact answer to the question has been obtained. [60, p. 137].

In view of these quotations, it is rather remarkable that dynamic pricing and learning has nowadays found its way in practice; many applications have been reported in various branches such as airline companies, the hospitality sector, car rental, retail stores, internet advertisement, and many more. A main cause for this is the fact that nowadays historical sales data is typically digitally available. This significantly reduces the efforts required to obtain sufficiently accurate estimates of the demand function. In addition, whenever products are sold via the Internet or using digital price tags, the costs associated with adjusting the prices in response to updated information or changed circumstances are practically zero. In contrast, a price-change in the pre-digital era would often induce costs, for example because a new catalog had to be printed or price tags had to be replaced. For a detailed study on such price-adjustment costs we refer to Zbaracki et al. [61] and the references therein. Slade [62] and Netessine [63] are two studies that consider dynamic pricing with costly price adjustments.

3. Dynamic pricing

In this section we discuss the literature on dynamic pricing. Because of the huge amount of literature on this subject, we cannot give a complete overview of the field. Instead, we briefly describe some of the major research streams and key references, in order to provide a context in which one can position the literature on dynamic pricing with learning discussed in Section 4. For a more elaborate overview of the vast literature on dynamic pricing, we refer to the books Talluri and van Ryzin [8], Phillips [9], Rao [12], Özer and Phillips [15], and the reviews by Bitran and Caldentey [6], Elmaghraby and Keskinocak [7], Heching and Leung [10], Gönsch et al. [11], Seetharaman [64], Chenavaz et al. [13] and Deksnyte and Lydeka [14].

The literature on dynamic pricing by a monopolist firm can roughly be classified as follows:

- Models where the demand function is dynamically changing over time.
- Models where the demand function is static, but where pricing dynamics are caused by the inventory level.

In the first class of models, reviewed in Section 3.1, the demand function changes according to changing circumstances: for example, the demand function may depend on the time-derivative of price, on the current inventory level, on the amount of cumulative sales, on the firm's pricing history, et cetera. In the second class of models, reviewed in Section 3.2, it is not the demand function itself that causes the price dynamics: a product offered in two different time periods against the same selling price is expected to generate the same amount of average demand. Instead, the price dynamics are caused by inventory effects, more concretely by changes in the marginal value of inventory. Naturally, it is also possible to study models that fall both in classes, where both the demand function is dynamically changing and the price dynamics are influenced by inventory effects; some of this literature is also reviewed in Section 3.2.

3.1. Dynamic pricing with dynamic demand

3.1.1. Demand depends on price-derivatives

Evans [65] is one of the first to depart from the static pricing setting introduced by Cournot [38]. In a study on optimal monopoly pricing, he assumes that the (deterministic) demand is not only a function of price, but also of the time-derivative of price. This models the fact that buyers do not only consider the current selling price in their decision to buy a product, but also anticipated price changes. The purpose of the firm is to calculate a price function, on a continuous time interval that maximizes the profit. The problem is solved explicitly using techniques from calculus of variations. Various extensions to this model are made by Evans [66], Roos [67–70], Tintner [71], and Smithies [72]. Thompson et al. [73] study an extended version of the model of Evans [65], where optimal production level, investment level, and output price have to be determined. Closely connected to this work is Simaan and Takayama [74], who consider a model where supply is the control variable, and where the time-derivative of price is a function of the current supply and current price. Methods from control theory are used to derive properties of the optimal supply path.

3.1.2. Demand depends on price history

A related set of literature considers the effect of reference prices on the demand function. Reference prices are perceptions of customers about the price that the firm has been charging in the past; see [75] for a review on the subject. A difference between the reference price and the actual selling price influences the demand, and, as a result, each posted selling price does not only affect the current demand but also the future demand. Dynamic pricing models and properties of optimal pricing strategies in such settings are studied, among others, by Greenleaf [76], Kopalle et al. [77], Fibich et al. [78], Heidhues and Köszegi [79], Popescu and Wu [80] and Ahn et al. [81].

3.1.3. Demand depends on amount of sales

Another stream of literature on dynamic pricing emerged from diffusion and adoption models for new products. A key reference is Bass [82], and reviews of diffusion models are given by Mahajan et al. [83], Baptista [84], and Meade and Islam [85]. In these models, the demand for products does not only depend on the selling price, but also on the amount of cumulative sales. This allows modeling several phenomena related to market saturation, advertisement, word-of-mouth effects, and product diffusion. Robinson and Lakhani [86] study dynamic pricing in such a model, and numerically compare the performance of several pricing policies with each other. Their work stimulated much further research on optimal dynamic pricing policies, see e.g. [87–89], and the references therein. The models studied in these papers are deterministic, and somewhat related to the literature following Evans [65]: both types of pricing problems are solved by principles from optimal control theory, and the optimal pricing strategy is often given by the solution of a differential equation.

Chen and Jain [90], Raman and Chatterjee [91], and Kamrad et al. [92] extend these models by incorporating randomness in the demand. In [90], the demand is determined by a finite-state Markov chain for which each state corresponds to a deterministic demand function that depends on price and cumulative sales. The optimal price path is characterized in terms of a stochastic differential equation, and compared with the optimal policy in a fully deterministic setting. Raman and Chatterjee [91] model uncertainty by adding a Wiener process to the (known) deterministic component of the demand function. They characterize the pricing policy that maximizes discounted cumulative profit, and compare it with the optimal price path in the fully deterministic case. Under some specific assumptions, closed form solutions are derived. Similar models that incorporate demand uncertainty are analyzed by Kamrad et al. [92]. For various settings they provide closed-form solutions of the optimal pricing policies.

3.2. Dynamic pricing with inventory effects

There are two important research streams on dynamic pricing models where the dynamics of the optimal pricing policy are caused by the inventory level: (i) ‘‘revenue management’’ type of problems, where a finite amount of perishable inventory is sold during a finite time period, and (ii) joint pricing and inventory–procurement problems.

3.2.1. Selling a fixed, finite inventory during a finite time period

In this stream of literature, a firm is assumed to have a certain number of products at its disposal, which are sold during a finite time period. There is no replenishment: inventory that is unsold at the end of the selling horizon is lost, and cannot be transferred to another selling season. In these problems, the dynamic nature of optimal prices is not caused by changes in the demand function, but by the fact that the marginal value of remaining inventory is changing over time. As a result, the optimal selling price is not a fixed quantity, but depends on the remaining amount of inventory and the remaining duration of the selling season.

Kincaid and Darling [93] may be the first to characterize and analyze the optimal pricing policy in such a setting. A more recent key reference is Gallego and van Ryzin [94]. They consider a continuous-time setting where demand is modeled as a Poisson process, with arrival rate that depends on the posted selling price. The pricing problem is formulated as a stochastic optimal control problem, and the optimal solution is characterized using the Hamilton–Jacobi–Bellman equation. For the exponential demand function a closed-form solution is derived. Because the optimal policy changes prices continuously, which may be undesirable in practical applications, two heuristics are proposed: one based on a deterministic version of the problem, and one based on determining the optimal fixed price. Both these heuristics are shown to be asymptotically optimal as the expected amount of sales grows large or as the length of the time horizon converges to zero. The authors further propose price heuristics – and show their asymptotic optimality – for many extensions of the problem: a discrete instead of a continuous set of feasible prices, customers who arrive according to a compound Poisson process, a demand function that depends both on price and on the time elapsed since the start of the selling season, the presence of holding costs and a discount rate, the case where initial inventory is a decision variable, and a setting where resupply, cancellations and overbookings are possible.

Numerous other extensions and variations of the model by Gallego and van Ryzin [94] have been studied: for example, settings with restrictions on the number of allowable prices or price changes [95–98], time-varying demand functions [99,100], and extensions to multiple stores [101] or multiple products [102,103] that share the same finite set of resources.

The extension to multiple products can be formulated as a dynamic program, which typically is intractable due to the curse of dimensionality. A number of papers study heuristic solutions [102, 104–109], which are typically based on static-price policies or on deterministic approximations or decompositions of the original dynamic program. Another stream of literature focuses on deriving structural properties of the price optimization problem, for various models of consumer demand [110–113].

Another important extension to Gallego and van Ryzin [94] is strategically behaving customers: customers who, when arriving at the (online) store, do not immediately decide whether to buy the product, but instead wait for a while to anticipate possible decreases in the selling price. In contrast, so-called myopic customers instantly decide whether to buy the product at the moment they arrive at the store. In such settings, the demand at a certain moment depends on the past, present, and future selling prices.

Dynamic pricing in view of strategic customers has received a considerable amount of research attention in recent years; a representative sample is Su [114], Aviv and Pazgal [115], Elmaghraby et al. [116], Liu and van Ryzin [117], Levin et al. [118], Cachon and Swinney [119] and Su [120]. Reviews of these literature are given by Shen and Su [121] and Gönsch et al. [122]. These studies typically have a game-theoretic flavor, since both the firm and the strategic customers have a decision problem to solve, with contradicting interests.

3.2.2. Jointly determining selling prices and inventory–procurement

A main assumption of the literature discussed above is that the initial capacity level is fixed. In many situations in practice this is a natural condition: the number of seats in an aircraft, rooms in a hotel, tables in a restaurant, or seats in a concert hall are all fixed for a considerable time period, and modifications in the capacity occur at a completely different time scale than dynamic price changes. In many other settings, however, the initial capacity is a decision variable to the firm; in particular, when the firm can decide how many items of inventory should be produced or purchased. Pricing and inventory management can then be considered as a simultaneous optimization problem.

This research field bridges the gap between the pricing and inventory management literature. Many different settings and models are subject to study, with different types of production, holding and ordering costs, different replenishment policies (periodic or continuous), finite or infinite production capacity, different models for the demand function, et cetera. Extensive reviews of the literature on simultaneous optimization of price and inventory decisions can be found in [123,124], [7, Section 4.1], [125–128].

4. Dynamic pricing and learning

In the static monopoly pricing problem considered by Cournot [38], the demand function is deterministic and completely known to the firm. These assumptions are somewhat unrealistic in practice, and eventually it was realized that demand should be modeled as a random variable. One of the first to pursue this direction is Mills [129], who assumes that the demand is the sum of a random term with zero mean and a deterministic function of price. He studies how a monopolist firm that sells finitely many products in a single time period should optimally set its production level and selling price. Further extensions of this model and properties of pricing problems with random demand are studied by Karlin and Carr [130], Nevins [131], Zabel [132], Baron [133,134], Sandmo [135] and Leland [136]. An important research question in these studies is how the firm's optimal decisions are influenced by the demand uncertainty and by the firm's attitude towards risk (risk-neutral, risk-averse, or risk-preferred).

The papers mentioned above model demand as a random variable, but still assume that the expected demand (as a function of the selling price) is completely known by the firm. This makes these models somewhat unrealistic and not usable in practice. The common goal of the literature on dynamic pricing and learning is to develop pricing policies that take the intrinsic uncertainty about the relation between price and expected demand into account.

In the next two sections we discuss the literature on dynamic pricing and learning. Section 4.1 considers the literature on the problem of a price-setting firm with infinite inventory and unknown demand function. This basically is the monopoly pricing problem described in Section 2.1, with uncertainty about the demand function. The full-information case of this problem is static; the price dynamics are completely caused by the fact that the firm learns about the price–demand relation through accumulating sales data. Section 4.2 discusses literature on pricing policies for firms selling a fixed, finite amount of inventory, with unknown demand function. For this problem, the full-information case is already dynamic by itself, as discussed in Section 3.2, and the learning aspect of the problem provides an additional source of the price dynamics.

4.1. No inventory restrictions

4.1.1. Early work

Uppsala econometrics seminar. The first analytical work on dynamic monopoly pricing with unknown demand curve seems to have been presented on August 2, 1954, at the 16th European meeting of the Econometric Society [137], by F. Billström, H. Laadi, and S.A.O. Thore, with contributions from L.O. Friberg, O. Johansson, and H.O.A. Wold. A mimeographed report Billström et al. [138] of the presented work has not been published in a journal, but an English reprint has appeared in [139,140]. These two works discuss the problem of a monopolist facing a linear demand curve that depends on two unknown parameters.

Thore [140] proposes to use a dynamic pricing rule that satisfies sign(pt − pt−1) = sign((pt−1 − pt−2)(rt−1 − rt−2)), where pt , rt denote the price and revenue in period t. This models the following intuition: if a previous price increase led to an increase in revenue, the price will again be increased; otherwise it will be decreased. Similarly, if a previous price decrease led to an increase in revenue, the price will again be decreased; otherwise, it will be increased. In addition, Thore [140] proposes to let the magnitude of the price adjustment depend on the difference between the last two revenue observations. He specifies two pricing rules in detail, and analyzes convergence properties of the resulting dynamical systems.

Billström and Thore [139] perform simulation experiments for one of these pricing rules, both in a deterministic demand setting and in a setting where a normally distributed disturbance terms added to the demand. They also extend the model to incorporate inventory replenishment, and provide a rule of thumb for the optimal choice of a constant appearing in the pricing rule.

Subsequent work. These studies emerging from the Uppsala Econometrics Seminar have not received much research attention in subsequent years. Clower [141] studies a monopolist firm facing a linear, deterministic demand function whose parameters may change over time. He discusses several price-adjustment mechanisms that may be applied by the firm to adapt its prices to changing situations. Baumol and Quandt [142] propose rules of thumb for the monopolist pricing problem, and assess their performance by a set of numerical experiments. In their Appendix A they propose a pricing rule equal to one of the rules proposed by Thore [140], although they are apparently unaware of that work. The authors investigate some convergence and stability properties of the resulting dynamical system, both in a discrete-time and continuous-time framework. Baetge et al. [143] extend the simulation results of Billström and Thore [139] to non-linear demand curves, and study the optimal choice of a constant appearing in the pricing rules. A final study in this line of research is from [144]. He studies a model where a monopolist has to decide on price, output level in the current period and maximum output in the next period. Three decision rules are compared with each other via a computer simulation. In addition, their performance is compared with a laboratory experiment, where test subjects had to determine their optimal pricing strategy.

4.1.2. Bayesian approaches

Several authors study the dynamic pricing and learning problem within a Bayesian framework.

Work by Aoki. One of the first is Aoki [145], who applies methods from stochastic adaptive control theory. He considers a setting where the demand function depends on unknown parameters, which are learned by the decision maker in a Bayesian fashion. The purpose is to minimize (a function of) the excess demand. He shows how the optimal Bayesian policy can, in theory, be computed via dynamic programming, but that in many situations no closed-form analytical expression of the solution exists. He then proposes two approximation policies. In the first, certainty equivalent pricing (CEP), at each decision moment the price is chosen that would be optimal if the current parameter estimates were correct. In the second, called an approximation under static price expectation, the firm acts at each decision moments as if the chosen price will be maintained throughout the remainder of the selling period. Aoki [146] shows that the prices generated by (two variants of) CEP and by the static-price-expectation approximation both converge a.s. to the optimal price.

Variations and extensions to Aoki [145]. An early study along the same lines of Aoki [145] is Chong and Cheng [147]. They assume a linear demand function with two unknown parameters and normally distributed disturbance terms, and formulate the optimal pricing problem as a Bayesian dynamic program. They show that certainty equivalent pricing is the optimal policy in case the slope of the demand function is known. For the case that both slope and intercept are unknown, they propose three approximations to the optimal policy: the approximation under static price expectation from [145], and two heuristics based on approximations of the value function. Their second approximation selects at each decision epoch the price that maximizes the difference between the expected profit and a term proportional to the covariance of the parameter estimates; this reflects the exploration–exploitation trade-off seen in many later work on optimization under uncertainty.

Other studies closely related to Aoki [145] are Nguyen [148,149], Wruck [150], Lobo and Boyd [151], Chhabra and Das [152], Qu et al. [153], and Kwon et al. [154]. Nguyen [148] considers a quantity-setting monopolist firm facing random demand in multiple periods, where the demand function depends on an unknown parameter which is learned by the firm in a Bayesian fashion. Structural (monotonicity) properties of the optimal policy are derived, and its performance is compared with a myopic one-period policy. Nguyen [149, Section 5] discusses similar questions in the context of a price-setting monopolist. Wruck [150] considers optimal pricing of durable and non-durable goods in a two-period model. The support of the uniformly distributed willingness-to-pay distribution is learned by Bayesian updating a uniform prior, and the optimal price policy is determined by solving a dynamic program. Lobo and Boyd [151] consider the same setting as Chong and Cheng [147], and compare via simulation the performance of four pricing policies with each other. Chhabra and Das [152] study the finite-time performance of standard multi-armed bandit policies and of a policy that (possibly incorrectly) assumes a linear demand function whose unknown parameters are learned by Bayesian updating of a Beta prior. Qu et al. [153] assume Bernoulli distributed demand with expectation a logit function of price, with a normal prior on the unknown parameters. Because this distributional form hampers exact calculation of posterior distributions, the authors discuss how to calculate an approximation. They propose a Bayesian-greedy price policy which cannot be computed exactly either, and show how an approximation can be calculated.

A numerical study compares the pricing policies with a few alternatives. Kwon et al. [154] study optimal markdown pricing in an infinite-horizon setting with discounted rewards, where the decision variables are the initial price, the markdown price, and the time of markdown. Cumulative demand is modeled as a Brownian motion with unknown drift which is either high or low and which is learned via Bayesian updating. The authors determine the optimal time of markdown, characterize the corresponding optimal initial and markdown prices, and prove a few monotonicity properties.

Finite action set. Some literature simplifies the problem by allowing only a finite set of feasible prices. This transforms the pricing-and-learning problem to a Bayesian multi-armed bandit problem. Leloup and Deveaux [155] show for Bernoulli distributed arms that approximations to Gittins-index policies circumvent the computational problems associated with solving the full Bayesian dynamic program. Wang [156] allows two feasible prices, assumes compound-Poisson distributed rewards from a certain parametric family, and approximate the Gittins index. Cope [157] considers a general Dirichlet prior on the discretized reservation-price distribution of customers, and develops approximations to the intractable Bayesian dynamic program that are closely related to Gittins-index policies. He shows that his pricing heuristics converge to the optimal price under an average-reward criterion, and argues that their performance do not suffer much from a misspecified prior distribution.

Incomplete learning and remedies. A common theme in the references mentioned above is that it is often intractable to compute the optimal Bayesian policy, and that therefore approximations are necessary. Rothschild [158] points to a more fundamental problem of the Bayesian framework. He assumes that there are only two prices the firm can choose, with demand for each price Bernoulli distributed with unknown mean. The dynamic pricing problem is thus viewed as a two-armed bandit problem. The optimal Bayesian policy can be computed via the corresponding dynamic programming formulation. The key result of Rothschild [158] is that, under the optimal Bayesian strategy, with positive probability the price sequence converges to a price that (with hindsight) is not the optimal price. McLennan [159] derives a similar conclusion in a related setting: the set of admissible prices is continuous, and the relation between price and expected demand is one of two known linear demand curves. It turns out that, under an optimal Bayesian policy, the sequence of prices may converge with positive probability to a price different from the optimal price. This work is extended by Harrison et al. [160], who show that in several instances a myopic Bayesian policy may lead to such ‘‘incomplete learning’’. They propose two modifications of the myopic Bayesian policy that avoid incomplete learning, and prove bounds on their performance.

Afèche and Ata [161] derive incomplete-learning results of similar flavor in the context of pricing different types of customers in an M/M/1 queue. Cheung et al. [162] extend Harrison et al. [160] to the case with $k \in N$ unknown demand functions, and where in addition at most m price changes are allowed. They propose a pricing policy, show that it achieves $\mathrm{Regret}(T) = O(\log(m)\, T)$ where $\log(m)$ denotes the m-times iterated logarithm, and prove that any policy has regret $\Omega(\log(m)\, T)$. Similar results are shown to hold in a continuous-time setting. Keskin [163] models the cumulative deviation from the expected demand at an incumbent price as the sum of a Wiener process and a drift-term proportional to the difference between selling price and incumbent price. The unknown drift-coefficient is learned by Bayesian updating of the parameters of a Gaussian prior. The author explains why a myopic policy induces incomplete learning, and characterizes the optimal learning policy as the solution to a partial differential equation (PDE). Based on this policy, he proposes a simple pricing rule that deviates from the myopic price proportionally to the squared coefficient of variation of the posterior belief on the optimal price, and that does not require solving a PDE. Numerical illustrations suggest that the performance of this heuristic is close to optimal.

Risk-averse pricing. All studies mentioned above assume that the firm is risk-neutral and optimizes the expected revenue. Sun and Abbas [164] depart from this assumption by studying the optimal price in a Bayesian dynamic-pricing-and-learning problem with a risk-averse seller. Choi et al. [165] study a family of simple pricing policies, in a Bayesian setting, based on separating the finite time horizon in an exploration and exploitation phase. They calculate the optimal risk-averse price with respect to a number of risk measures, and provide numerical examples.

Economics and econometrics literature. The economics and econometrics literature also contains several studies on pricing and Bayesian learning. Prescott [166], Grossman et al. [167], Mirman et al. [168] consider two-period models and study the necessity and effects of price experimentation. Trefler [169] focuses on the direction of experimentation, and applies his results on several pricing problems. Rustichini and Wolinsky [170] and Keller and Rady [171] consider a setting where the market environment changes in a Markovian fashion between two known demand functions, and study properties of optimal experimentation.

Balvers and Cosimano [172] consider a dynamic pricing model where the coefficients of a linear demand model change over time, and discuss the implications of active learning. Willems [173] aims to explain observed discreteness in price data. The author considers a model where the expected demand depends linearly on price via two time-varying parameters with known expectation. The author elaborates a Bayesian learning approach via dynamic programming, and discusses the differences between active and passive learning. Easley and Kiefer [174], Kiefer and Nyarko [175], Aghion et al. [176] are concerned with Bayesian learning in general stochastic control problems with uncertainty. They study the possible limits of Bayesian belief vectors, and show that in some cases these limits may differ from the true value. This implies that active experimentation is necessary to obtain strongly consistent control policies.

Related studies on optimal market design. Finally, we mention the studies Manning [177] and Venezia [178] on optimal design of market research. Manning [177] considers a monopolist firm facing a finite number of customers. By doing market research, the firm can ask n potential customers about their demand at some price p. Such market research is not for free, and the main question of the paper is to determine the optimal amount of market research. This setting is closely related to pricing rules that split the selling season in two parts (e.g. the first pricing rule proposed by Witt [144]): in the first phase, price experimentation takes place in order to learn the unknown parameters, and in the second phase of the selling season, the myopic price is used. Venezia [178] considers a linear demand model with unknown parameters, one of which behaves like a random walk. The firm learns about these parameters using Bayes' rule. In addition, the firm can learn the true current value of this random walk by performing market research (which costs money). The author shows how the optimal market-research policy can be obtained from a dynamic program.

4.1.3. Non-Bayesian approaches

Policies without performance bounds. Despite the disadvantages of the Bayesian framework outlined above (computational intractability of the optimal solution, the results by Rothschild [158] and McLennan [159] on incomplete learning), it has taken several decades before research on pricing policies in a non-Bayesian setting took off. An early exception is Aoki [146], who proposes a pricing scheme based on stochastic approximation in a non-Bayesian framework. He proves that the prices converge almost surely to the optimal price, and compares the policy with Bayesian pricing schemes introduced in [145]. More recently, Carvalho and Puterman [179,180] and Morales-Enciso and Branke [181] propose several pricing heuristics based on approximations of an underlying finite-horizon dynamic program whose states contain historical price/demand observations (reminiscent of the dynamic-programming approximations developed by Aoki [145,146]).

Seminal paper with performance bounds. A disadvantage of the many pricing heuristics that have been proposed in the literature, both in a Bayesian and a non-Bayesian setting, is that a qualitative statement of their performance is often missing. In many studies the performance of pricing policies is only evaluated numerically, without any analytical results. This changes with the groundbreaking work of Kleinberg and Leighton [182], who quantify the performance of a pricing policy by Regret(T): the expected loss in T time periods incurred by not choosing optimal prices. They consider a setting where buyers arrive sequentially to the firm, and buy only if their willingness-to-pay (WtP) exceeds the posted price. Under some additional assumptions, they show that if the WtP of the individual buyers is an i.i.d. sample of a common distribution, then there is no pricing policy that achieves Regret(T) = o(√T); in addition, there is a pricing policy that achieves Regret(T) = O(√T log T). In an adversarial or worst-case setting, where the WtP of individual buyers is not assumed to be i.i.d., they show that no pricing policy can achieve Regret(T) = o(T^2/3), and that there is a pricing policy with Regret(T) = O(T^2/3 (log T)^1/3).

Parametric approaches. Le Guen [183], Broder and Rusmevichientong [184], den Boer and Zwart [185], den Boer [186], and Keskin and Zeevi [187] take a parametric approach, using classical maximum (quasi)likelihood or least-squares estimators to estimate unknown parameters of the demand function. Le Guen [183] considers a multi-product setting with linear demand, assuming a particular structure on the unknown parameter matrices. He shows that certainty equivalent pricing augmented with price experimentation at predetermined time intervals leads to prices converging to the true optimal price, and proposes a pricing heuristic for non-linear demand functions. Assuming a single product setting with Bernoulli distributed demand, Broder and Rusmevichientong [184] show a $\sqrt{T}$ lower bound on the regret, using information-theoretic inequalities and techniques found in [188], and show that a pricing policy that strictly separates exploration from exploitation achieves $O(\sqrt{T})$ regret. This rate can be improved to $O(\log T)$ if the demand-function is such that there are no ''uninformative prices'': prices at which the expected demand given a (erroneous) parameter estimate is equal to the true expected demand (the existence of such prices plays an important role in the best achievable growth rate of the regret; cf. [160]). Den Boer and Zwart [185] consider an extended class of generalized linear single-product demand models, and show in an example that certainty equivalent pricing is not strongly consistent. They propose a pricing policy that always chooses the price closest to the certainty equivalent price that guarantees a minimum amount of price dispersion. This price dispersion, measured by the sample variance of the selling prices, guarantees convergence of the prices to the optimal price, and leads to $\text{Regret}(T) = O(T^{1/2+\delta})$, for arbitrary small $\delta > 0$. Den Boer [186] extends this policy to multiple products, attaining $\text{Regret}(T) = O(\sqrt{T} \log T)$ for so-called canonical link functions and $\text{Regret}(T) = O(T^{2/3})$ for general link functions. Keskin and Zeevi [187] assume a linear demand function, show a $\sqrt{T}$ lower bound on the regret using proof techniques different from [184], and generalize Broder and Rusmevichientong [184] and den Boer and Zwart [185] by providing sufficient conditions for any pricing policy to achieve $\text{Regret}(T) = O(\sqrt{T} \log T)$. Assuming that the mean demand is exactly known at a certain price, they show that $\text{Regret}(T) = O(\log T)$ is attainable. Both these results are extended to the multiple-product setting, focusing on a class of so-called orthogonal pricing policies.

Robust optimization. Eren and Maglaras [189] study dynamic pricing in a robust optimization setting. They show that if an infinite number of goods can be sold during a finite time interval, it is optimal to use a price-skimming strategy. They also study settings where learning of the demand function occurs, but under the rather strong assumption that observed demand realizations are without noise. Bergemann and Schlag [190,191] and Handel et al. [192] also consider pricing in a robust framework, but their setting is static instead of dynamic. Handel and Misra [193] consider a two-period model where a monopolist sets prices based on a set of demand curves feasible with acquired sales data. The authors describe and analyze the optimal two-period pricing policy that minimizes a dynamic version of the minimax regret, and investigate how customer preferences influence the difference between dynamic and static prices.

Finite action set. If the demand model is assumed to lie in a finite set of known demand functions, the dynamic-pricing-and-learning problem can be regarded as a multi-armed bandit problem with dependent arms. This viewpoint is taken by Tehrani et al. [194], who develop a pricing policy based on the likelihood-ratio test, and show that its regret is bounded assuming that there are no uninformative prices. Their work can be viewed as a non-Bayesian counterpart to Harrison et al. [160] and Cheung et al. [162].

Variants. Pricing without demand information in a queueing model is studied by Haviv and Randhawa [195]. They consider the problem of pricing delay-sensitive customers in an unobservable M/M/1 queue. The purpose of the paper is to study the impact of ignoring arrival-rate information on the optimal pricing strategy. The authors find that a policy that ignores this information performs surprisingly well, and in some cases can still capture 99% of the optimal revenue.

Finally, we mention Jia et al. [196], who consider dynamic pricing and learning in an electricity market, where the goal is to steer the expected demand to a desired level. This particular objective is reminiscent of the multi-period control problem discussed in [197,198]. A stochastic-approximation type policy inspired by Lai and Robbins [199] is shown to achieve $O(\log T)$ regret, and in addition it is shown that no policy can achieve sub-logarithmic regret. The fact that logarithmic instead of $\sqrt{T}$ regret can be achieved is caused by subtle differences between dynamic pricing and this multi-period control problem, which are further discussed in [185, Remark 1].

4.2. Finite inventory

We here discuss the literature on dynamic pricing and learning in the presence of finite inventory that cannot be replenished. Most of the studies assume a finite selling season, corresponding to the models discussed in Section 3.2.1. Some studies assume an infinite time horizon, and consider the objective of maximizing total discounted reward.

4.2.1. Early work

Lazear [200] considers a simple model where a firm sells one item during at most two periods. In the first period a number of customers visit the store; if none of them buys the item, the firm adapts its prior belief on the value of the product, updates the selling price, and tries to sell the item in the second period. The author shows that the expected profit increases by having two selling periods instead of one. He extends his model in several directions, notably by allowing strategic behavior of customers who may postpone their purchase decision if they anticipate a price decrease. Sass [201] extends the model of Lazear and studies the relation between the optimal price strategy and the number of potential buyers.

4.2.2. Bayesian approaches

Unknown arrival rate, known willingness-to-pay. Aviv and Pazgal [202] start a research stream on Bayesian learning in dynamic pricing with finite inventory. Customers arrive according to a Poisson process with unknown arrival rate, and purchase a product with (known) probability $\exp(-p)$, where p is the current selling price. The unknown arrival rate is learned via Bayesian updates of a Gamma prior. The authors characterize the optimal continuous-time pricing policy by means of a differential equation. Because this equation does not always admit an explicit solution, three pricing heuristics are proposed: certainty equivalent pricing (CEP), a fixed price policy, and a naive pricing policy that ignores uncertainty on the market. Numerical experiments suggest that CEP performs quite well. Lin [203] considers a similar setting, allowing for general willingness-to-pay distributions. He proposes a pricing policy where the seller sets the price based on repeatedly updated estimates of the demand distribution, and evaluates its performance via simulations.

Araman and Caldentey [204], Farias and van Roy [205] and Mason and Välimäki [206] study the infinite-horizon discounted-reward case. Araman and Caldentey [204] assume a known willingness-to-pay distribution and a two-point prior distribution on the unknown arrival rate, propose a pricing heuristic based on an asymptotic approximation of the value function of the underlying optimal-control problem, and compare its performance numerically with CEP, static pricing, and a two-price policy. In a similar setting Farias and van Roy [205] take a finite mixture of gamma distributions as prior, propose a pricing heuristic called decay balancing, and show numerically that it frequently outperforms CEP and the heuristics of Araman and Caldentey [204]. They further show that the expected discounted reward obtained from decay balancing is at least one third of the optimal reward, and discuss an extension to multiple stores. Mason and Välimäki [206] assume that only a single item is sold, with either high or low customer arrival-rate which is learned in a Bayesian fashion. The authors study structural properties of the optimal price policy and compare it to policies that neglect learning.

Avramidis [207] observes that the number of arrivals or the number of sales is sufficient to compute the posterior arrival-rate distribution, given any prior. This means that the setting considered by Aviv and Pazgal [202], Lin [203], Araman and Caldentey [204] and Farias and van Roy [205] can be resolved without imposing a specific family of priors (Gamma, two-point discrete, finite mixture of Gammas).

Unknown arrival rate and unknown willingness-to-pay. Chen and Wang [208] consider pricing of a single asset in an infinite horizon with discounted rewards, and assume that the willingness-to-pay distribution is unknown but equal to one of two known distributions. The authors formulate a Bayesian dynamic program and prove that the optimal prices decline over time if the hazard rates of these distributions can be ranked uniformly; a counterexample shows that this condition cannot be relaxed to first-order stochastic dominance. Sen and Zhang [209] extend the model of Aviv and Pazgal [202] by assuming that the purchase probabilities of arriving customers are not known to the firm. They assume that the demand distribution is an element of a finite known set, and consider a discrete-time setting with Bayesian learning and a gamma prior on the arrival rate. The optimal pricing policy can be explicitly calculated, and, in an extensive computational study, its performance is compared with both a full information setting and a setting where no learning occurs.

Partially observable Markov decision processes. Aviv and Pazgal [210] consider a Markov-modulated demand environment, modeled by a Markov chain where each of the finite states corresponds to a different known demand function, and where the state of the system is learned in a Bayesian fashion. The optimal pricing problem is formulated as a Partially Observable Markov Decision Process, that turns out to be computationally intractable. Various approximate solutions are proposed that rely on modifying the information structure of the problem, and their performance is evaluated in a numerical study. Chen [211] considers a similar partially observable Markov decision process. In his setting, the seller estimates the willingness-to-pay distribution of customers based on two-sided censored observations. Three near-optimal price heuristics are proposed and their performance is assessed by numerical experiments. For exponentially or Weibull distributed demand, more refined results on the behavior of the heuristics are obtained.

4.2.3. Non-Bayesian approaches

Asymptotic regime where inventory grows large. The optimal pricing problem studied by Gallego and van Ryzin [94] does often not admit an explicit solution, and the authors therefore consider an asymptotic regime where both the demand and the level of inventory grow large. They prove that the optimal revenue obtained in this asymptotic regime serves as an upper bound for the optimal expected revenue of the original problem, and show that the optimal asymptotic pricing policy is to use a static price throughout the sales horizon. This static price is the maximum of the unconstrained optimal price (the revenue-maximizing price in the case of infinite inventory) and of the clearance price (the price that induces a stock-out precisely at the end of the sales horizon).

Besbes and Zeevi [212] initiate a stream of literature that attempts to learn this optimal static price in an incomplete-information setting. For both a parametric and non-parametric setting they develop pricing policies, based on the idea of dividing the sales horizon into an ‘‘exploration’’ phase during which the demand function is learned and an ‘‘exploration’’ phase during which the perceived optimal price is used. To establish performance bounds, they consider a sequence of problems indexed by $n \in \mathbb{N}$, where the n-th problem has initial inventory nx and demand function $n\lambda$, for some $x > 0$ and some function λ. They prove an $O(n^{-1/4}(\log n)^{1/2})$ upper bound on the relative regret of this policy in the non-parametric setting, an $O(n^{-1/3}(\log n)^{1/2})$ bound in the parametric setting, and an $O(n^{-1/2}(\log \log n)(\log n)^{1/2})$ bound in case the demand function is known up to a single unknown parameter.

These upper bounds are complemented by results showing that all policies have relative regret $\Omega(n^{-1/2})$. In a non-parametric setting, Wang et al. [213] improve these upper bounds by developing a policy with relative regret $O(n^{-1/2}(\log n)^{4.5})$; apart from the logarithmic term, this policy thus achieves the asymptotically optimal regret rate. The pricing policy is based on the idea of iterative price experimentation in a shrinking series of intervals that with high probability contain the optimal price. Lei et al. [214] provide a further improvement by even removing the logarithmic terms in the upper bound: they propose and analyze three pricing algorithms based on ideas from bisection search and stochastic approximation, and show that one of these algorithms achieves $O(n^{-1/2})$ relative regret.

Besbes and Zeevi [215] extend Besbes and Zeevi [212] to a setting where multiple products share the same finite resources. They show that a policy which separates ''exploration'' and ''exploitation'' achieves relative regret $O(n^{-1/(3+d)}(\log n)^{1/2})$, where d is the number of products. This is improved to $O(n^{-1/(3+d/s)} (\log n)^{1/2})$ if one assumes additional smoothness conditions on the demand function, including s-times differentiability, and to $O(n^{-1/3}(\log n)^{1/2})$ if the set of feasible prices is discrete and finite; this last setting is accompanied by an $\Omega(n^{-1/2})$ lower bound on the relative regret of any policy. Another variant of the problem is studied by Besbes and Maglaras [216], who study the situation where certain financial milestone constraints in terms of sales and revenues targets are imposed. They formulate a pricing policy that periodically updates its pricing decisions in order to the most stringent financial constraint, and show an $O(n^{-1/2}(\log n)^{1/2})$ bound on the relative regret. Avramidis [207] modifies the type of policy proposed by Besbes and Zeevi [212] by estimating both arrival rate and purchase probabilities, and by using the solution of the finite-time Markov decision problem in the exploitation phase instead of the solution that corresponds to the asymptotic regime. Numerical experiments suggest that these modifications lead to lower regret.

Alternative asymptotic regimes. The high-volume regime of the studies mentioned above is not applicable for settings with relatively low inventory. This motivates den Boer and Zwart [217] to study a setting with multiple, consecutive, finite selling seasons and finite inventories. They show in a parametric framework that this setting satisfies a certain endogenous-learning property, which implies that price experimentation is hardly necessary to eventually learn the optimal prices. The authors prove that the (cumulative) regret of a small modification of certainty equivalent pricing is $O(\log^2 T)$ after T selling seasons, and that any policy has regret $\Omega(\log T)$.

In [218], a firm tries to sell k items to n potential customers. The authors propose an online pricing-and-learning policy that does not require parametric assumptions on the demand distribution, and that exploits the fact that the problem is closely related to multi-armed bandit problems (despite the finite inventory, which means that the optimal price depends on the current inventory level). They show a regret bound of $O((k \log n)^{2/3})$, and provide a matching lower bound on the regret. If the ratio k/n is sufficiently small this bound is improved to $O(\sqrt{k} \log n)$.

Bertsimas and Perakis [219] consider pricing in a least-squares-learning setting with a single selling season. They formulate a dynamic program that describes the optimal pricing policy but is computationally intractable, and propose approximate solutions based on state-space reductions.

For an extended model with competitors and slowly varying parameters, estimation methods and price policies are discussed. No performance bounds on pricing policies are provided.

Robust optimization. A number of studies take a robust approach, where the demand function is not learned over time but assumed to lie in some known uncertainty set. Lim and Shanthikumar [220] and Thiele [221] study this in a single-product setting, and Lim et al. [222], Thiele [223] in a multi-product setting. With these robust approaches no learning takes place, despite the accumulation of sales data. Cohen et al. [224] develop an approach to dynamic pricing and learning that attempts to bridge this gap between robust and data-driven approaches, by sampling different scenarios from historical sales data. A robust extension of Araman and Caldentey [204] and Farias and van Roy [205], where finite inventory is sold during an infinite time horizon, is studied by Li et al. [225]. Another approach that does not rely on historical demand data is Xiong et al. [226] (see also [227–229]). They model demand uncertainty using fuzzy set theory, propose different fuzzy programming models, and present an algorithm based on fuzzy simulation and a genetic algorithm to solve these problems. Dziecichowicz et al. [230] do not focus on determining optimal prices, but instead study the optimal timing of markdowns. The authors derive a robust optimization problem to determine the optimal markdown policy, which in some special cases can be solved exactly and in other cases can be approximated by a mixed-integer problem. The results are also extended to multiple products. Finally, Ferrer et al. [231] assume that demand is a non-random function of price, and lies in an uncertainty set of demand functions known to the decision maker. The authors introduce a measure of risk aversion, and study the relation between risk-aversion and properties of the optimal pricing policy, both theoretically and by numerical simulations.

Variants. Gallego and Talebian [232] consider a setting where a finite number of products is offered in multiple versions during a finite sales horizon. Demand is modeled by a customer choice model. The different product-versions share an unknown ‘‘core value’’, which is estimated by maximum likelihood estimation. The possible time-varying arrival rate of customers is learned in Bayesian fashion. The authors develop a pricing rule in a rolling horizon framework, and illustrate its behavior by a computational study. Somewhat related to this is Berg and Ehtamo [233], where a firm sells different versions of a product to different customer segments. The utility functions of each segment are partly unknown, and learned by the firm using stochastic gradient methods or variants thereof.

4.3. Machine-learning approaches

A considerable stream of literature on dynamic pricing and learning has emerged from the computer science community. In general, the focus of these papers is not to provide a mathematical analysis of the performance of pricing policies, but rather to design a realistic model for electronic markets and subsequently apply machine learning techniques. An advantage of this approach is that one can model many phenomena that influence the demand, such as competition, fluctuating demand, and strategic buyer behavior. A drawback is that these models are often too complex to analyze analytically, and insights on the behavior of various pricing strategies can only be obtained by performing numerical experiments.

Machine-learning techniques that have been applied to dynamic pricing problems include evolutionary algorithms [234, 235], particle swarm optimization [236], reinforcement learning and Q-learning [237–252], simulated annealing [253], Markov chain Monte Carlo methods [254], the aggregating algorithm [255] by Vovk [256], goal-directed and derivative-following strategies in simulation [257,258], neural networks [259–263], and direct search methods [259,264–266].

These papers all use very different models, methods, assumptions and performance metrics. This makes it hard to compare different papers with each other, which we therefore do not attempt.

4.4. Joint pricing and inventory problems

A few studies consider the problem of simultaneously determining an optimal pricing and inventory replenishment policy while learning about the demand.

Parametric approaches. Most of them consider learning in a Bayesian framework. Subrahmanyan and Shoemaker [267] assume that the unknown demand function lies in a finite known set of demand functions, and is learned in a Bayesian fashion. The optimal policy is determined by solving a dynamic program. Several numerical experiments are provided to offer insight in the properties of the pricing policy. Bitran and Wadhwa [268] and Bisi and Dada [269] study a similar type of problem, where an unknown parameter is learned in a Bayesian manner, and where the optimal decisions are determined by solving a dynamic program. Bitran and Wadhwa [268] perform extensive computational experiments, and Bisi and Dada [269] derive several structural properties of the optimal policy. Zhang and Chen [270] consider Bayesian learning of a component of the demand distribution that does not depend on the selling price, and show that the finite-horizon expected discounted profit is maximized by a so-called base stock list price policy [271,124]. Motivated by industrial practice of fashion retailers, Choi [272] considers a scenario where the seller can order inventory and change selling prices at two distinct stages in the selling season. Information obtained from the first stage is used in a Bayesian manner to determine optimal decisions in the second stage. The author formulates a dynamic program, proves several structural properties, and carries out numerical experiments to illustrate his results. Gao et al. [273] formulate and solve a two-period two-products pricing and inventory problem with Bayesian learning of unknown demand parameters. Forghani et al. [274] allow for a single price change, formulate the optimal control problem, and show numerical examples. In [275], a manufacturer estimates the optimal capacity decision from advance sales information obtained prior to the regular selling season. Formal learning of demand parameters is not considered, but the electronic companion to the paper elaborates upon an extension to Bayesian learning.

Non-parametric and robust approaches. Burnetas and Smith [276] consider a joint pricing and inventory problem in a non-parametric setting. They propose an adaptive stochastic-approximation policy, and show that the expected profit per period converges to the optimal profit under complete information. A robust approach to the dynamic pricing and inventory control problem with multiple products is studied by Adida and Perakis [277]. The focus of that paper is the formulation of the robust optimization problem, and to study its complexity properties. Related is the work of Petruzzi and Dada [278]. These authors assume that there is no demand noise, which means that the unknown parameters that determine the demand function are completely known once a demand realization is observed that does not lead to stock-out.

Adida and Perakis [279] discuss several robust and stochastic optimization approaches to joint pricing and procurement under demand uncertainty, and compare these approaches with each other in a numerical study. Mahmoudzadeh et al. [280] formulate a robust control problem for joint pricing and production in a hybrid manufacturing/remanufacturing system, where the coefficients of a linear demand function are unknown. Arasteh et al. [281] consider a similar setting for a joint pricing-and-inventory problem.

Variants. Lariviere and Porteus [282] consider the situation of a manufacturer that sells to a retailer. The manufacturer decides on a wholesale price offered to the retailer, and the retailer has to choose an optimal inventory replenishment policy. Both learn about a parameterized demand function in a

Bayesian fashion. Properties of the optimal policy, both for the manufacturer and the retailer, are studied. Gaul and Azizi [283] assume that a product is sold in different stores. The problem is to determine optimal prices in a finite number of periods, as well as to decide if and how inventory should be reallocated between stores. Parameters of the demand function are learned by Bayesian updating, and numerical experiments are provided to illustrate the method.

5. Methodologically related areas

Dynamic pricing under uncertainty is closely related to multi-armed bandit problems. This is a class of problems that capture many essential features of optimization problems under uncertainty, including the well-known exploration–exploitation trade-off: the decision maker should properly balance the two objectives of maximizing instant reward (exploitation of current knowledge) and learning the unknown properties of the system (exploration). This trade-off between learning and instant optimization is also frequently observed in dynamic pricing problems.

The literature on multi-armed bandit problems is large; some key references are Thompson [284], Robbins [285], Lai and Robbins [286], Gittins [287], Auer et al. [288]; see further Vermorel and Mohri [289], Cesa-Bianchi and Lugosi [290], and Powell [291]. If in a dynamic pricing problem the number of admissible selling prices is finite, the problem can be modeled as a classical multi-armed bandit problem. This approach is e.g. taken by Rothschild [158], Xia and Dube [253], and Cope [157]. If the set of admissible selling prices is a continuum, then the dynamic pricing problem is closely related to the continuum-armed bandit problem. This problem has recently been studied, among others, by Kleinberg [292], Auer et al. [293], Cope [294], Wang et al. [295], Goldenshluger and Zeevi [296], Rusmevichientong and Tsitsiklis [297], Filippi et al. [298], Abbasi-Yadkori et al. [299], Yu and Mannor [300], Slivkins [301], Perchet and Rigollet [302], Combes and Proutiere [303]. Pricing and learning in a time-varying market, as discussed in Section 6.3, is related to the non-stationary multi-armed bandit problem [304,305].

Another important area related to dynamic pricing and learning is the study of convergence rates of statistical estimates. Lai and Wei [306] study how the speed of convergence of least-squares linear regression estimates depend on the amount of dispersion in the explanatory variables. Their results are applied in several dynamic pricing problems with linear demand functions, such as Le Guen [183] and Cooper et al. [307]. Similarly, results on the convergence rate of maximum-likelihood estimators, as in [308,309], are crucial in the analysis of pricing policies by Besbes and Zeevi [188], Broder and Rusmevichientong [184] and den Boer and Zwart [185].

Dynamic pricing and learning can also be put in a general framework of stochastic control problems with parametric uncertainty. For various dynamic economic models, such problems have been considered by Easley and Kiefer [174], Kiefer and Nyarko [175], Marcet and Sargent [310], Wieland [311,312], Beck and Wieland [313], Han et al. [314], amongst many others. Kendrick et al. [315] review some of this literature, focusing on continuous-time optimal control problems with linear system equations, quadratic cost functions, and Gaussian additive noise terms.

Clearly, many sequential optimization problems in operations research that are studied in an online-learning setting have methodological similarities with dynamic pricing and learning: inventory control ([316–318], is just a small sample of the vast literature), online advertisement (e.g. [319]), resource allocation (e.g. [320]), assortment planning (e.g. [321–323]), product launch and product exit optimization (e.g. [324]), et cetera. These problems can also be studied in conjunction with optimal pricing. For example, Talebian et al. [325] study simultaneous pricing and assortment optimization with Bayesian demand learning, and Section 4.4 lists a number of studies that combine pricing and inventory control in an incomplete-information setting.

6. Extensions and new directions

Most of the literature discussed above studies dynamic pricing and learning for a monopolist firm that sells a single product to nonstrategic customers in a stationary market environment. Here we review literature on a number of extensions to this setting: strategic customer behavior, competition, time-varying markets, and model misspecification. Note that the first three of these extensions can in some sense be regarded as extending the literature of Section 3.1, on dynamic pricing with dynamic demand, to an incomplete-information framework.

6.1. Strategic consumer behavior

The importance of incorporating the effect of strategically behaving customers on the dynamic pricing policy has repeatedly been recognized; cf. the papers on dynamic pricing with strategic customer behavior mentioned in Section 3.2.1. The study of dynamic pricing with both strategic customer behavior and demand learning, however, is not that well developed.

Learning of the willingness-to-pay distribution. Studies in which the demand distribution is not assumed to be known are Loginova and Taylor [326], Levina et al. [255], Weaver and Moon [327] and Caldentey et al. [328], and Lazear [200] which is discussed in Section 4.2.1. In [326], the seller learns the willingness-to-pay in a stylized model with a single customer whose product valuation is either ‘‘high’’ or ‘‘low’’. The authors characterize game-theoretic equilibria and study the effect of strategic behavior of the buyer. Levina et al. [255] consider a seller who learns about a complicated demand process that incorporates, among other things, strategic customer behavior. This behavior is described using a game-theoretic consumer choice model. The focus of the paper is on properties and numerical performance of an online-learning algorithm proposed by the authors. Weaver and Moon [327] and Caldentey et al. [328] take a robust-optimization approach. Inspired by sales of food and agriculture products, Weaver and Moon [327] formulate a robust pricing model with multiple products, multiple customer segments, and linear demand function, and numerically compare its performance with two ‘‘price-assurance policies’’: a policy that offers refund to consumers if the price falls below their purchase price, and a policy that ensures that price will not decline sufficiently to induce refunds. Caldentey et al. [328] consider a robust formulation of the seller’s pricing problem for a discrete-time infinite-inventory setting with unknown valuations and arrival times of customers, and characterize the pricing policies that minimize the worst-case regret, both in case of myopic or strategic customers. The authors also analyze the limit of the regret and optimal pricing strategies as the length of the discrete time periods goes to zero.

No learning of the willingness-to-pay distribution. In [329–331], the unknown quality of the product is learned. This introduces a game with strategic behavior of customers, who may postpone their purchase in order to retrieve more information, induce a markdown, or influence the belief of the seller about the quality of product. In [332] consumers learn about the capacity of the seller in repeated two-period selling season, and in [333] consumers learn their valuations while the firm learns the market size. Mersereau and Zhang [334] considers a firm that learns the relative amount of strategic customers. None of these studies involve learning the demand or willingness-to-pay distribution (given the quality of the product).

6.2. Competition

Several studies address the issue of incorporating competition in dynamic pricing and learning. This is challenging, because one generally does not know how competitors behave, and their behavior influences a firm’s own optimal pricing policy. The literature references in this section are not comprehensive, but meant to give a flavor of available approaches and results.

Parametric approaches. The already mentioned paper by Bertsimas and Perakis [219] considers a least-squares learning setting with finite inventories, multiple competitors, and linear demand functions with slowly varying parameters. The authors propose an estimation and price-optimization scheme, and illustrate the method by numerical examples. The related study Kachani et al. [335] considers multiple quantity-setting firms selling the same type of products with finite inventories. The price–demand relation is linear, with unknown parameters that may slowly vary over time. The authors formulate the optimization problem as a mathematical program with equilibrium constraints, provide some computational results, and derive a closed-form expression of a limiting Nash equilibrium in case of two firms and a selling horizon of two time periods. Kwon et al. [336] formulate a finite-time finite-capacity oligopolistic pricing problem in the language of differential variational inequalities. They consider a demand model where the time-derivative of demand is proportional to the difference between a firm's price and a moving average of prices used in the past. Unknown model parameters are estimated with a Kalman filter, which is recalculated at several moments in the sales horizon. The authors propose an algorithm to solve the resulting differential variational inequalities, and illustrate their method by a numerical example. A similar setting is considered by Li et al. [337] and Chung et al. [254], who incorporate a non-parametric functional-coefficient autoregressive time series in their demand model. Duopoly models are studied, among many others, by Coughlan and Mantrala [338] in a Bayesian framework and by Choi and Jagpal [339] in a setting with risk-averse firms.

Robust approaches. A robust optimization approach to dynamic pricing in an oligopolistic environment with demand uncertainty is taken by Perakis and Sood [340] (see also the closely related study Friesz et al. [341]). They consider a multi-period setting with fixed, finite inventory levels, and study Nash equilibrium price policies. Adida and Perakis [342] consider a joint pricing-and-inventory control problem with duopolistic competition and demand uncertainty, and formulate it as a robust optimization problem. The authors discuss existence and uniqueness of a Nash equilibrium and address various computational aspects of the problem.

Economics and econometrics literature. A large stream of economics literature is concerned with the long-term behavior of price adjustment processes in oligopolistic settings, cf. [343–357]. One usually assumes that firms are using a certain specific learning scheme, and then studies whether the selling prices converge to a Nash equilibrium. While these papers are interesting from the perspective of understanding price formation, application in business practice is hampered by the fact that one generally does not know which learning scheme competing firms are using. A remarkable early study on pricing and learning in a competitive environment is Barta and Varaiva [358], who study convergence properties of various stochastic-approximation based price-adjustment rules.

Computer science literature. Contributions from the computer science community to dynamic pricing with demand uncertainty in a competitive environment include Greenwald and Kephart [359], Dasgupta and Das [360], Tesauro and Kephart [361], Kutschinski et al. [237], Könönen [239], Jumadinova and Dasgupta [362,363].

6.3. Time-varying market parameters

Recently several studies have appeared that depart from the strong assumption of unchanging market conditions. We here only list papers that include mathematical performance analysis of the proposed pricing rules.

Besbes and Sauré [364] study the classical finite-inventory finite selling-season pricing problem in a setting with deterministic demand functions. The seller anticipates that the initial demand function d may at some moment τ change to $d^\theta \in \{d^1, \ldots, d^K\}$, where (τ , θ) are random variables with known distributions and where $d^1, \ldots, d^K$ are known functions. The authors formulate the optimal pricing problem and derive conditions that induce monotonicity properties of the optimal pricing strategy. Chen and Farias [365] also study pricing in a finite-inventory setting where the market process is fluctuating over time. They propose a pricing heuristic which is based on the idea of frequently re-calculating the estimated optimal fixed price. The authors prove bounds on the performance loss relative to an optimal clairvoyant pricing strategy in an asymptotic regime where inventory and demand grows large.

In an infinite-inventory setting, Besbes and Zeevi [188] study a pricing problem where customers' willingness-to-pay (WtP) distribution changes at some unknown point in time. The WtP distribution before and after the change is assumed to be known, only the time of change is unknown to the seller. Lower bounds on the worst-case regret are derived, and pricing strategies are developed that achieve the order of these bounds. Keskin and Zeevi [366] consider a linear demand function with time-varying parameters, which are estimated with weighted least squares. They measure the amount of change in the parameter process by the quadratic variation over T periods, and assume it is bounded by BT^{ν}, for some known $\nu \in [0, 1]$ and unknown $B \geq 0$. An $\Omega(T^{(2+\nu)/3})$ lower bound on the regret attainable by any policy is proven, and a pricing policy that separates the time horizon into pure exploration and exploitation periods is shown to have regret $O(T^{(2+\nu)/3})$. More refined expressions are derived in case B is known. If only ''bursty'' changes occur, the authors show that a regret of $O(T^{1/2} \log T)$ can be achieved. Den Boer [367] considers a similar set-up with linear demand and weighted least-squares estimation, but assumes that only the size of the market (the intercept) is unknown. A myopic pricing policy is considered, and upper bounds on the long-term average regret are provided. The methodology is illustrated by applying it to pricing in the Bass model, and to pricing in a competitive environment.

A particular structural form of the evolution of the market process is assumed by Wang et al. [368] and Chakravarty et al. [369]. Wang et al. [368] assume that the demand intensity over the life-cycle of an information good behaves as the density function of a Weibull distribution, whose unknown parameters are learned in a Bayesian fashion using Monte-Carlo simulation techniques. Chakravarty et al. [369] propose a simulation-based algorithm to calculate the optimal price path for a product-diffusion model introduced by Robinson and Lakhani [86].

6.4. Model misspecification

Studies that assume a parametric form of the demand functions face the risk of model misspecification. Besbes and Zeevi [370] show in a single-product setting that the loss incurred by such misspecification may in some settings be not that significant. Another form of misspecification is incorrectly assuming that there are no competitors present. Schinkel et al. [354], Tuinstra [357], Bischi et al. [371,372], Isler and Imhof [373], Cooper et al. [307], Anufriev et al. [374] study the effect of this error on the resulting equilibria in various (linear) models, elaborating on earlier work by Kirman [344,375–377]. In an airline revenue-management setting, Cooper et al. [378] show that incorrectly assuming high-fare and low-fare class passengers behave independently can be detrimental for the firm's revenue.

7. Conclusion

Dynamic pricing with incomplete demand information is a topic that has received considerable research attention in recent years. Different scientific communities have studied characteristics of pricing policies, usually with different aims in mind: the operations research/management science literature typically focuses on finding an optimal pricing policy from the perspective of a seller, and on proving optimality properties in tractable models; the economics literature is generally more concerned with explaining price behavior and price formation observed in markets; and the computer science literature is typically not afraid to consider complicated demand models that are not tractable for mathematical analysis, but that can be handled using appropriate machine learning techniques. In spite of these different aims, there is much in overlap in the studied demand models, proposed pricing policies, and the techniques deployed to analyze the behavior of these policies. With this survey we aim to provide a comprehensive overview of these studies, and facilitate further research on this broad and lively research topic.

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Appendix 2: Review of the demand price elasticity for water – experience from Greek cities

Price elasticity of demand measures the responsiveness of demand to a change in price. This principal is based on consumers' attempt to optimize their utility function.

The investigation of the researches conducted for the estimation of demand price elasticity for water in Greece indicated few projects with reliable estimation of mentioned indicator. The research differ by size of the agglomeration, sample of the investigated inhabitants and research period. The review was made with the differentiation of the calculation starting from average prices and marginal prices (the average price is defined as the water bill paid by the consumer divided by the amount of water consumed. The marginal price is the price that a consumer should pay, according to the water price structure, for the next m^3 of water). Review of identified projects is summarised below.

1. Value -0,55: -0,24 – data estimated from 4 regional cities: Xanthi, Komotini, Ioannina, Alexandroupolis. Different tariffs' across the cities (flat rate/block tariffs') . Research in period 2003-2005. Daily water consumption pc 115-130l. Source: Gratziou M., Andreadaki M., Tsalkatidou M., Water demand and rates policy in provincial cities in Greece in: European Water 15/16 pg 33-44, 2006. www.ewra.net/ew/pdf/EW_2006_15-16_04.pdf
2. Value: -0,42; -0,433. data for one city: Kozani. Research in period 2005-2012. Increasing block tariffs (8 blocks). Source: Kanakoudis V., Gonelas K., Forecasting the Residential Water Demand, Ballancing Full Water Cost Pricing and Non-RevenueWater Reduction Polices. Procedia Engineering 89 (2014) 958 – 966. http://www.sciencedirect.com/science/article/pii/S1877705814026459
3. Value: -0,1. Data for Athens Metropolitan Area. Research in period 2000-2010. Source: Bithas Kostas, Stoforos Chrysostomos, Estimating Urban Residential Water Demand Determinants and Forecasting Water Demand for Athens Metropolitan Area, 2000-2010 South-Eastern Europe Journal of Economics 1 (2006) 47-59.
4. Value: -0,19;-0,72. Thessaloniki. Research in period 1994-2000. Source: Mylopoulos Y., Kolokytha E., Mentes A., Vagiona D., Urban water demand management – The city of Thessaloniki-Greece case study. In: Advances in Water Supply Management. Butler D., Menon F.,Maksimovic C., Sweets&Zeitlinger, Lisse 2003. https://books.google.pl/books?id=SCUT4QDVkJEC&pg=PA721&lpg=PA721&dq=demand+price+elasticity+water+greece&source=bl&ots=gGA9IbOPwG&sig=oFGZJ0yr_UbkXv8zcEEpW6a24mw&hl=pl&sa=X&ved=0CFwQ6AEwB2oVChMI0Zmby7vwxgIVQlcUCh3xagIh#v=onepage&q=demand%20price%20elasticity%20water%20greece&f=false

Taking into account that the projects no 3 and 4 are related to very big agglomeration the results from the first two are much more applicable to the case of Skiathos. The preliminary assumption that the most appropriate value ranges at -0,4 can be applied for the model, however the elasticity of the output on the other values will be verified.