The project HAMACSIS has studied the effects of Lie group actions in several relevant problems in the areas of symplectic and related geometries, as well as in dynamical systems. The motivation for this study was that the symmetries of geometric structures (implemented by smooth actions of Lie groups) is a powerful tool for their study, since it allows to treat all the points that are related by a symmetry as equivalent, therefore replacing the geometric space under study by a smaller one. This process is called “reduction”, and the smaller spaces obtained by identifying equivalent points, reduced spaces. From the point of view of dynamics, most mechanical systems evolve on phase spaces equipped with geometrical structures that are preserved by the symmetry of a Lie group action. In this case the study of these symmetries gives us important information about the behaviour of the system and the qualitative properties of its solutions.

This motivation is foundational for a long tradition of efforts in geometry and dynamics since the early works of Lagrange and Hamilton. In more recent years, the existence of groups of symmetries with singularities has emerged as an important topic within the area. Singularities happen when one point in our geometric space stays fixed by the action of some subset of the group of symmetries. This implies several difficulties in the implementation of the program described above at a number of levels. Notably, the reduced spaces obtained after the identification of equivalent points no longer form a smooth manifold, but a singular topological space (technically, a stratified space) that cannot be studied with the usual tools of differential geometry. In addition, the existence of these singularities can deeply influence the qualitative behaviour of dynamical systems preserved by the group of symmetries, and these singularities are responsible, to a large extent, for a profound change in the stability and bifurcations of the solutions. This project tried to formalize and study in a systematic way these effects of the singularities of Lie group actions in different situations.

The results obtained in this project can be summarized in the following: The effects of symmetries and their singularities in the description of several singular geometric spaces of interest in mathematical physics and geometry have been studied in [2] [3] and [7]. In [7] we have obtained a description of the reduced space corresponding to a regularized singular action on a cotangent bundle. Cotangent bundles are the most important case of geometric spaces that act as universal phase spaces for any classical or quantum mechanical system. In that reference, we have provided a picture for the topology and Poisson geometry of their reduced space in the particular case that the symmetric action exhibits a single type of singularity (hence the “regularization” in the action). The relationship of the geometry of the reduced spaces with the structure of mechanical systems with internal structure (such as classical particles in a Yang-Mills gauge field) has also been studied. In [2], in preparation, these results are extended to the general case of arbitrary symmetry actions without a prescribed number of singularities. In [3] the same questions have been investigated for another important type of geometric spaces which recently were found to be extremely relevant in many areas of geometry and mathematical physics, Lie groupoids. In that reference we give a concrete characterization of the global stratified structure of the reduced spaces.

In [1] [4] [6] and [8] the effects of singularities of symmetric actions in several kinds of dynamical and mechanical systems have been investigated. In [4], we have explicitly shown how the singularities can affect the stability properties of an important family of solutions of Hamiltonian systems known as relative equilibria, which correspond to physically observed steady motions of a mechanical system, like uniformly rotating bodies. We have shown how the previous methods that fail in an identification and systematic use of these singularities can offer less optimal stability criteria than our main result. In [1], in preparation, this approach is used towards the more ambitious task of developing a unified theory of bifurcations of Hamiltonian relative equilibria that exhibit continuous groups of singularities within their symmetric actions. These methods have been used in [8], where the geometric techniques developed were employed in order to study the stability of several cases of Riemann Ellipsoids (a classical model for planetary and stellar evolution) were the existence of singularities are relevant. We have shown that the power of these methods allows to improve, in a simple way, the classical stability results obtained by Cartan, Lyapounov and Poincare, among others.

Finally, in [6], yet another geometric formalism for mechanical systems is studied from the point of view of its symmetries. We study the effect of continuous symmetries on the existence of constants of motion for Lagrangian mechanical systems on Lie algebroids. This is relevant since Lagrangian systems on Lie algebroids provide a very general framework that encompasses several previously unrelated classes of dynamical systems, like classical mechanical, Euler-Poincare or Lagrange-Poincare systems. In [6] we prove a Noether-like theorem for Lie algebroids, which provides a way to obtain constants of motion from each one-parameter group of symmetries of the system. In addition, we also study how additional constants of motion can be found for systems on Lie algebroids admitting alternative Lagrangian formulations.

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