2 Publishable summary

2.1 Summary description of the project objectives

The project *Lax algebras in homotopy theory* proposes to develop the theory of lax algebras in the context of homotopy. The former exploits a monadic approach to topological structures that allows for a simultaneous treatment of both *neighborhood* and *convergence* structures: the filter monad $\mathbb{F} = (F, \eta, \mu)$ on the category Set of sets and set-maps yields a presentation of a topological space either as a set X equipped with a map $\nu : X \to FX$ that associates to an element of X its neighborhood system, or as a set X equipped with a relation $c \subseteq FX \times X$ that specifies which filters converge to which points. This formalized perspective opens up unexplored venues between topological theories and categorical homotopy or higher-dimensional categorical concepts.

The monadic and higher-dimensional categorical aspects of lax algebras will be explored in relation with homotopical themes while keeping in mind the guiding objective: the development of a homotopy theory for lax algebras.

2.2 Description of the activities carried out since the beginning of the project

In the context of the research project, the grant beneficiary has obtained promising results, and is currently working on further developments with an array of colleagues. In particular, with R. Lucyshyn-Wright he has written a new chapter in the monograph *Monoidal Topology*, and is completing existing chapters in collaboration with the other authors of the volume (M.M. Clementino, E. Colebunders, D. Hofmann, R. Lowen, C. Schubert and W. Tholen). He is regularly taking part in the topology seminar at the host institute and has attended and participated in a variety of scientific meetings; he has given 6 oral presentations in Swizerland, Belgium, and France at internal seminars or conferences.

In parallel with his research activities, the grant beneficiary has taught a 1-semester math course at the master level, has developed and is currently teaching two 2-semester math courses for 1st year students in architecure at the host institute. He has also supervised 4 semester-long student research projects at the host institute.

2.3 Description of the main results achieved so far

Shortly before the start of the project, the grant recipient's work *Order-adjoint monads and injective objects* (J. Pure Appl. Algebra 214, 2010) demonstrated the power of the monadic approach to topology by describing the injective objects of lax algebras. As a by-product, he obtained both classical and novel results: for instance, continuous lattices form the initially determined injective objects of the category Top of topological spaces and continuous maps, frames play the same role in the category $\mathsf{Cls}_{\mathrm{fin}}$ of finitary closure spaces, as do sup-semilattices in the category Met of generalized metric spaces. Since injectivity is an important theme in homotopy theory, this work also laid out the groundwork for the current project.

In *On the monadic nature of categories of ordered sets* (to appear in *Cah. Topol. Géom. Différ. Catég.*), the grant recipient takes the previous work one step further and demonstrates how monadic structures are implicit in categories of lax algebras. In particular, the category of lax algebras for which the associated Eilenberg-Moore category describes the initially determined injective objects is the category that carries the most significant monadic information, while the category of preordered sets and monotone maps underlies all categories of lax algebras, and consequently all associated Eilenberg-Moore categories. The main theorem

Theorem. *If $\tau_X : \mathbb{S} \to \mathbb{T}$ is a monad morphism from an order-adjoint monad $\mathbb{S}$, then there is an isomorphism of Eilenberg-Moore categories that is identical on morphisms:*

$$\mathsf{Set}^{\mathbb{T}} \cong \mathsf{Mon}(\mathsf{Set}_{\mathbb{S}})^{\mathbb{T}'} ,$$

where $\mathbb{T}'$ is the monad induced by the forgetful functor $\mathsf{Set}^{\mathbb{T}} \to \mathsf{Mon}(\mathsf{Set}_{\mathbb{S}})$.

allows a systematic treatment of monadicity results, gathering particular instances scattered throughout the literature with new ones under a common roof. This result also suggests potential developments with *directed* homotopy theory.

2.4 Expected final results and their potential impact

The expected results of the project pertain to exponentiability in categories of lax algebras, as well as the development of enriched aspects of the theory of lax algebras in the context of homotopy-theoretic themes. Potential developments for directed homotopy theory are aimed at, as well as results for descent theory.

The impact of the project naturally depends on the results of forthcoming investigations. However, in the wake of the early successes of the project, the theory of monads is expected to ascertain its role a central tool in the development of unifying models in topological settings.

Contact

Scientific coordinator: Kathryn Hess Bellwald <kathryn.hess@epfl.ch>
MathGeom Institute, Epf Lausanne, 1015 Switzerland.
Phone: +41 21 693 42 45. Fax: +41 21 693 03 85.

Grant recipient: Gavin J. Seal <gavin.seal@epfl.ch>
MathGeom Institute, Epf Lausanne, 1015 Switzerland.
Phone: +41 21 693 03 86. Fax: +41 21 693 03 85.