

Summary report

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Smooth Ergodic Theory and Partially Hyperbolic Diffeomorphisms

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Different processes that appear in many areas of science - Physics, Mechanics, Astronomy, Chemistry, Biology - can be modeled by dynamical systems: the movement of the planets around the sun, a chemical reaction, the evolution of the population of some species. In all such systems there is a 'state space', the set of all possible states of the system, and a 'law of change' which tells us what will be the next state of the system if we know the present one - this can be either a function, if we consider discrete time, or an ordinary differential equation (ODE) which produces a flow, if we consider continuous time. Instead of finding explicit formulas for the solutions of these systems, which may be very complicated or even impossible, and may not give much insight into the behavior of the system, the researchers in Dynamics are interested in the qualitative properties of the solutions, or trajectories: if they are periodic, or converge to some limit, or are chaotic, how often they visit some given regions of the space, what happens with nearby trajectories or if we perturb the system, etc., and this new area of Mathematics, Dynamical Systems, became a very active field of research. Like mathematicians in other areas, researchers in dynamics would like to understand all (or at least most of) these systems: divide them into groups of systems with similar properties, understand which properties are compatible within the same system and which are not, which properties would imply other properties, find invariants which could characterize some class of systems or differentiate two systems. Even if we do not study only the concrete systems coming from other areas of science, our results and expertise can always be applied to any new systems that may arise. The main objective of the project was the study of different relationships between some local aspects of dynamics, things that happen at the level of points or subsets, and some global aspects, or properties that can be seen at large scales, even the entire space, and are usually stable under perturbations.

During the grant period we investigated different problems related to this aspects. I wrote with J. Llibre a survey on these topics even before the start of the grant (this was one year after the submission of the proposal). Also with J. Llibre we analyzed the relationship between the topological entropy (an invariant measuring the complexity of the map) and the periodic orbits for maps on graphs, and we found a criteria for chaotic behavior: the existence of a periodic orbit with the period having a large enough odd divisor. We also investigated when the topological index at a fixed point (an invariant characterizing the dynamics near the fixed point, important because of its good topological properties) of a map or flow can be determined just by looking at a finite number or partial derivatives. We gave some sufficient conditions for this to happen, and we proved that this is the case for most systems. We investigated if a system proposed by E. Lorenz in meteorology is integrable (the trajectories can be given implicitly), and we concluded that it is not: it has only two analytic independent first integrals, and not four - this is an indication of complicated (maybe chaotic) behavior. With E. Vargas I studied the invariant measures for flows on surfaces. For an interesting class of flows (Cherry flows), we showed that the invariant measures (describing globally the statistics of the trajectories) depend in fact on the derivative of the vector field at a fixed point (a local property). With Z. Xia I studied the closed currents supported on stable and unstable foliations, and we concluded that they are unique up to rescaling and have nonzero homology for transitive Anosov maps, and they are boundaries for Anosov flows and other specific cases. Also the nonzero homology of the unstable foliation implies in some cases the integrability of the center-stable bundle. More details about these results is contained in the report on work progress and achievements.

We wrote these results in several papers: *On Dirac physical measures for transitive flows*, together with W. Sun and E. Vargas, published in 2010 in 'Communications in Mathematical Physics' (this paper was started before the beginning of the grant), *Topological entropy and periods for graph maps*, with J. Llibre, accepted and published on-line in 2011 in 'Journal of Difference Equations and Applications', *The index of singularities of vector fields and finite jets*, with J. Llibre, published in 2011 in 'Journal of Differential Equations', *On the analytic integrability of the 5-dimensional Lorenz system for the gravity-wave activity*, with J. Llibre, submitted for publication, and *Invariant measures for Cherry flows*, with E. Vargas, available online at the arXiv mathematical repository. I also wrote the paper *Results and open questions on some invariants measuring the dynamical complexity of a map*, with J. Llibre, published in 2009 (before the grant period) in 'Fundamenta Mathematicae', which was one of the objectives of the proposal, and I am currently writing two other papers, one alone and one in collaboration with Z. Xia.

The results we obtained were also disseminated during presentations at seminars at Universitat Autònoma de Barcelona, Universitat Politècnica de Catalunya, Instituto de Matemática i Estadística - Universidade de São Paulo, De Paul University in Chicago, Université Paris-Sud Orsay, as well as conferences like the AIMS Conference in Dynamical Systems, Differential Equations and Applications in Dresden, 2010. We received a good feedback, and our work was appreciated, both by the referees of the journals where we published the papers, and by the members of the scientific community with whom we discussed or who attended our presentations.

A significant amount of time was dedicated to the training activities. During this period I gained expertise and I contributed with my research in different areas of Dynamical Systems like One-dimensional Dynamics, Singularity Theory, Integrability of ODE. I participated at 15 conferences, workshops, meetings and advanced schools in Dynamical Systems in Spain, France, Germany, United Kingdom, Poland, United States and Brazil. I participated actively at the weekly seminars in Dynamical Systems from the Barcelona area, I visited Northwestern University, Chicago, USA and Universidade de São Paulo, Brazil, and I collaborated with other scientists, both from Barcelona and other places.

The research on the topics of the proposal is ongoing, and will certainly continue in the future, because it has a considerable potential for further development. For example, the work on the invariant measures for Cherry flows could be extended to generalized Cherry flows, while the remaining case should be treated in the future. Likewise, the results obtained on finding the index of a fixed point could produce an algorithm which could be used in order to compute this index numerically. Also the research on the homology associated with the stable and unstable foliations should be continued in the future, the uniqueness up to rescaling of the closed stable and unstable currents for transitive Anosov maps should have some interesting consequences, while the connection between nonzero homology and integrability could be extended to other situations. We also hope that the collaborations established during this period with researchers like Jaume Llibre, Edson Vargas, Zhihong Xia, Graham Smith, Kamlesh Parwani, will continue to be fruitful in the future.

In conclusion, we believe that the Marie Curie grant had a positive impact both on the researcher, giving him the chance to advance in his career, by increasing his expertise in Dynamical Systems, collaborating with other mathematicians, and carrying on high quality research, as well as on the European and international scientific community in general, through the results obtained and published by the researcher and his collaborators. I would like to thank the European Commission and CRM Barcelona for giving me this opportunity.