Publishable summary (final report)

One of the deepest unsolved problems of the representation theory of finite groups is Brauer's $k(B)$ problem posed in 1959 which is to show that the number $k(B)$ of complex irreducible characters in a p-block B is at most the size of a defect group of B. In 1962 Nagao showed that for p-solvable groups this problem is equivalent to the so-called $k(GV)$ problem which is the following. Let V be a finite faithful FG-module for some finite field F and finite group G. Suppose that $|G|$ is coprime to $|F|$. Form the semidirect product GV of V by G and denote the number of conjugacy classes in GV by $k(GV)$. The $k(GV)$ problem is to show that $k(GV) \leq |V|$. The $k(GV)$ problem was eventually solved in 2004 by combined efforts of many mathematicians. The proof is approximately 500 journal pages long.

One of the aims of this project was to make steps in generalizing the $k(GV)$ theorem. The new problem under consideration was the non-coprime $k(GV)$ problem proposed in 2004 by Guralnick and Tiep. Given a finite faithful completely reducible FG-module V for some finite field F and finite group G. In most cases, is it true that $k(GV) \leq |V|$? This would shed more light on Brauer's $k(B)$ problem as well.

Just like the classical $k(GV)$ theorem, a possible, future solution of the non-coprime $k(GV)$ problem would break into three parts: quasisimple case, extraspecial case, and the imprimitive case. In this four year period Guralnick and the researcher have essentially completed the extraspecial case of the non-coprime $k(GV)$ problem. It has been shown that, in the extraspecial case, there are only finitely many groups GV for which $k(GV) > |V|$. In particular, it has been proved, in the extraspecial case, that $k(GV) \leq \max\{|V|, 2^{120}\}$.

In the course of the work on the previous problem two related projects have also been successfully undertaken.

Let G be a finite group, F a field, and V a finite dimensional FG-module such that G has no trivial composition factor on V. Guralnick and the researcher showed that the arithmetic average dimension of the fixed point spaces of elements of G on V is at most $(1/p)\dim V$ where p is the smallest prime divisor of the order of G. This answers and generalizes a 1966 conjecture of Neumann which also appeared in a paper of Neumann and Vaughan-Lee and also as a problem in The Kourovka Notebook posted by Vaughan-Lee. This result also generalizes a recent theorem of Isaacs, Keller, Meierfrankenfeld, and Moretó. Various applications have been given. For example, another conjecture of Neumann and Vaughan-Lee has been proven and some results of Segal and Shalev have been improved and/or generalized concerning BFC groups.

The other problem having a similar flavor to the non-coprime $k(GV)$ problem has been to bound the minimal base size of a finite linear group.

For a finite permutation group $H \leq \mathrm{Sym}(\Omega)$ a subset of the finite set Ω is called a base, if its pointwise stabilizer in H is the identity. The minimal base size of H (on Ω) is denoted by $b(H)$. The minimal base size $b(G)$ of a finite linear group G in its natural action on the underlying vector space is defined in the same way. Halasi and the researcher, in a soon to be submitted paper, have proven the following. Let V be a finite vector space over a finite field of order q and characteristic p. Let $G \leq GL(V)$ be a p-solvable linear group with $O_p(G) = 1$. Then $b(G) \leq 2$ unless $q \leq 4$ in which case $b(G) \leq 3$. The first statement extends a recent deep and complicated result of Halasi and Podoski and the second statement generalizes a celebrated theorem of Seress.

A finite simple group can be generated by two elements. In fact, Guralnick and Kantor have shown that if x is an arbitrary non-identity element of a finite simple group G then there exists y in G so that x and y together generate G. Therefore it is interesting to study the so-called generating graph $\Gamma(G)$ of a finite group G defined on the non-identity elements of G with an edge between x and y if and only if they generate G. The second part of this four year Marie Curie project was to obtain results on the generating graph. One of the highlights of such investigations has been a work of Breuer, Guralnick, Lucchini, Nagy, and the researcher on Hamiltonian cycles in the generating graph. The focus of this research has been the following question. Does $\Gamma(G)$ contain a Hamiltonian cycle if and only if $|G| \geq 4$ and for every non-trivial normal subgroup N of G the factor group G/N is cyclic. This problem has been solved for solvable groups. It has also been shown that if G is a sufficiently large simple group or a sufficiently large symmetric group then $\Gamma(G)$ contains a Hamiltonian cycle.

Another motivation for the definition of the generating graph came from so-called covering problems in group theory. To illustrate the obtained results belonging in the third part of this Marie Curie project, normal coverings will be discussed. A covering of a group G is a set of proper subgroups of G whose union is the whole of G. A covering is normal if every conjugate of every subgroup in the covering is also in the covering. The minimal size of a normal covering of a finite non-cyclic group G is denoted by $\gamma(G)$. There are many papers on this invariant. For example, Bubboloni, Praeger, and Spiga have shown that $\gamma(S_n)$ is bounded from above and below by positive constant multiples of n where S_n denotes the symmetric group of degree n. At the same time Britnell and the researcher considered the analogues problem of bounding $\gamma(\mathrm{GL(n,q)})$ where $\mathrm{GL(n,q)}$ denotes the general linear group of n by n matrices over the field of q elements. It not only has been shown that $\gamma(\mathrm{GL(n,q)})$ is bounded from above and below by positive constant multiples of n but it has been proved, among other results, that there exists an explicit lower bound of n/π^2 for the minimal size of a normal covering of the group $\mathrm{GL(n,q)}$.

Finally, there were a few other works funded by this four year project which do not fall in any of the above three main categories as far as the results are concerned however the methods used to prove these do. Only one such publication will be discussed here.

Sidki called a finite group G expansive if for any normal set R of G and any conjugacy class C of G the normal set RC contains at least as many conjugacy classes of G as R does. This notion originates from a paper of Sidki published in the early 1980's. The problem is to classify all expansive groups. Bezerra, Halasi, Sidki, and the researcher have managed to show that a finite group G is expansive if and only if it is a direct product of expansive simple and abelian groups. Hence Sidki's problem reduces to a problem on simple groups. It has also been shown that many simple groups are indeed expansive.

For more information see `http://www.renyi.hu/groups.html`.