Summary description of the project objectives

My research relates to the area of algebraic geometry, a branch of modern pure mathematics, and it includes some aspects of tropical geometry, a combinatorial piece-wise linear geometry. In this project, I study the geometry of Severi varieties on toric surfaces in arbitrary characteristic. I address four main problems, as follows:

Problem 1 (*Dimension problem)* Find the dimension of $V_{\Sigma,\mathscr{L},g}$.

Problem 2 (*Geometry of curves)* Describe the geometry of a general curve C of genus g in $|\mathscr{L}|$, in particular classify the singularities of C.

Problem 3 (*Enumeration of curves)* Find enumerative formulas for the number of curves of genus g in $|\mathscr{L}|$ satisfying certain (linear) constraints, e.g., passing through a collection of given points in general position.

Problem 4 (*Irreducibility problem)* Find examples of toric surfaces admitting reducible/irreducible Severi varieties $V^{irr}_{\Sigma,\mathscr{L},g}$ in positive characteristic. Classify the toric surfaces Σ for which $V^{irr}_{\Sigma,\mathscr{L},g}$ are irreducible. Prove the irreducibility of Severi varieties $V^{irr}_{\Sigma,\mathscr{L},g}$ on toric surfaces in characteristic zero.

The work performed since the beginning of the project

Transfer of knowledge: I gave two graduate courses on Algebraic Geometry, organized students seminar for several terms, an international workshop, and have been organizing a weekly seminar on Algebraic Geometry and Number Theory for the last five years. I also attended several international conferences, and gave many talks on the topic of the project at the conferences and at several seminars in Brazil, France, Germany, Israel, Switzerland, Turkey, and US.

Research: I have achieved most of the objectives and technical goals of the project with natural deviations due to the development of my research. The main results are described below.

The main results

I proved that $\dim V^{irr}_{\Sigma,\mathscr{L},g} = -K_\Sigma.\mathscr{L} + g - 1$ in arbitrary characteristic, which solves the Dimension Problem (problem 1). This generalizes a famous theorem of Zariski, who proved the result in the case of plane curves in characteristic zero. Similar results were proved in the characteristic zero case by Caporaso, Harris, Arbarello, Cornalba, Vakil, myself and others, but none of the proofs could be generalized to the case of positive characteristic! In this project, I have developed a completely new approach, that allows to obtain the result in arbitrary characteristic. This result appeared in my paper *On Zariski's theorem in positive characteristic* in the Journal of EMS (Vol. 15, No. 5, (2013) pp. 1783–1803). I also proved a stronger version of Zariski's theorem for rational surfaces in characteristic zero. The proof has been published as an appendix to the paper of Kleiman and Shende *On the Göttsche threshold*, which appeared in Clay Mathematics Proceedings (Volume 18, pp. 429–449, Appendix: pp. 442–447).

In the Geometry of Curves Problem (problem 2), I have classified the singularities of general rational curves on toric surfaces. As expected, the singularities are either nodes or have type A_r, where $r+1$ is divisible by the characteristic in the case of odd characteristic, and slightly more complicated in the case of characteristic two. The observation that a general curve of a given genus on a toric surface may have singularities other than nodes is true only in positive characteristic. This phenomenon was unexpected by the mathematical community, and it shows that the second part of the famous Zariski's theorem mentioned above can not be generalized to the case of positive characteristic. Furthermore, I have shown that in positive characteristic, there may exist a curve such that any curve of a given genus intersects it non-transversally. This is also a new phenomenon that does not occur in characteristic zero. Some of these results appear in my paper in the journal of EMS.

The results of my investigation of the Irreducibility Problem (problem 4) were surprising and unexpected. First, as expected, I found examples of toric surfaces admitting reducible Severi varieties in positive characteristic, which appeared in my paper in the journal of EMS. In my examples, different components of Severi varieties parameterize curves with different types of singularities. Next, I analyzed the examples deeper, and found examples of toric surfaces admitting reducible Severi varieties in *any genus and arbitrary characteristic,* including characteristic zero! One of these examples will appear in the proceedings of Gökova Geometry and Topology Conference 2013. This discovery changed the direction of my research in the Irreducibility Problem, and opened many new questions, including classification question in characteristic zero. In the case of elliptic curves, I was able to prove that (apart from several explicitly given exceptions) the Severi variety of curves of genus one on a toric surface Σ is reducible if and only if Σ is a quotient of another toric surface Σ' by a finite subgroup of the torus, which acts freely on the one-dimensional orbits of Σ'. This result and the current state of the art will appear in my paper *On (ir)reducibility problem for Severi varities on toric surfaces* (in preparation). The complete classification of toric surfaces admitting reducible Severi varieties for curves of higher genera in characteristic zero is an exciting new problem I continue to work on.

Finally, very recently, I was able to prove tropical enumerative formulae for counting curves with certain multiplicities of a given genus on a toric surface that pass through an appropriate amount of points in general position, which gives a solution to the Enumeration of Curves Problem (problem 3). The multiplicity assigned to an algebraic curve is defined in terms of the moduli spaces of stable maps from the normalization of the curve to the toric surface. If the curve is nodal then the multiplicity is one, and hence in characteristic zero this count coincides with the usual count. However, if the curve is not nodal (as may happen in positive characteristic) then the multiplicity is strictly greater than one. In the proof I generalized the methods I used in my paper *Tropical geometry and correspondence theorems via toric stacks*, and used new technical ingredients based on the results of Abramovich, Olsson, and Vistoli.

Potential impact and use of the results

The results of the project deepen the link between algebraic and tropical geometries, significantly improve the understanding of the geometry of Severi varieties and the knowledge of positive-characteristic phenomena that have not been previously studied. They generalize and extend some of the famous results of Zariski, Harris, Vakil, Mikhalkin, and others. I am certain that the techniques developed in this project will be useful in other problems in algebraic and enumerative geometry, singularity theory, mirror symmetry etc. All together, it contributes to European excellence and competitiveness.

The work on this project and its results have lead to new questions and research problems, and produced new collaborations. In particular, this year I started a new joint project with Shustin (Israel), H. Markwig (Germany), and T. Markwig (Germany) in which we study tropicalization, realization, and algebraic-tropical correspondence problems, which is closely related to the topic and the methods of my IRG-supported research. The new project is supported by the German-Israeli Foundation.

Regarding broader impact and applications, let me mention that it is a common problem in pure mathematics that it is hard to predict the applications that the research will have for other aspects of human activity. Moreover, when applications are found, they are usually based on theories that had been developed in dozens, or even hundreds, of previous studies. Today, there is a clear realization that mathematics that would have been considered as purely theoretical 40 years ago is finding applications in physics, biology, statistics, and industry. In particular, algebraic geometry is currently being applied in cryptography and physics, while tropical geometry can be used in algebraic statistics and computational biology.