Our aim has been to understand further the solutions of the three-dimensional incompressible Navier–Stokes equations (3DNSE)

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \qquad \nabla \cdot u = 0, \qquad u(0) = u_0.$$

Whether or not these equations possess smooth solutions that exist for all time for any smooth initial condition is one of the Clay Foundation's Million Dollar Millennium Prize Problems.

Most of our work during the project period has been based on work of Fursikov from the 1970s, treating not the Navier–Stokes equations themselves but the equations for the evolution of the moments of an initial measure.

The remarkable thing about this approach is that while the Navier–Stokes equations themselves are nonlinear, the moment equations are linear. One can illustrate this fact very simply in the scalar ordinary differential equation

$$\dot{x} = -x + x^2 \tag{ODE}$$

(a model to which we will return), if one takes an initial measure μ on $\mathbb{R}$ and considers the evolution of its moments

$$m_k = \int x^k \, \mathrm{d}\mu.$$

Then

$$\dot{m}_k = \int k x^{k-1} \dot{x} \, \mathrm{d}\mu = -k m_k + k m_{k+1},$$

and the equation has become linear. Despite this simplification, note that the equation for m_k requires knowledge of m_{k+1}; this is the same difficulty that arises in trying to solve the moment equations that arise from the Navier–Stokes equations themselves.

In a series of papers, Fursikov developed an approach to solving the moment equations that arise from the Navier–Stokes equations: let us write them formally as

$$\frac{\partial M_k}{\partial t} = A M_k + B M_{k+1} \qquad M_k(0) = f_k(0). \tag{$*$}$$

It is known that there is a correspondence between the smoothness of solutions of the Navier–Stokes equations and smoothness of solutions of the moment equations, *provided that the moments arise from a measure* (we will return to this later).

Fursikov's approach was to consider a penalised minimisation problem

$$\text{minimise } J(\underline{M}, \underline{m}) = \frac{\epsilon}{2} \|\underline{M}\|_X^2 + \frac{1}{2} \|\underline{m} - \underline{f}\|_Y^2$$

subject to the constraint that $\underline{M} = \{M_k\}$ satisfies $(*)$ but with $M_k(0) = m_k(0)$. (Here X and Y are appropriate space, such that if $\underline{M} \in X$ then the moments are regular, and if $\underline{M} \in X$ then $\underline{M}(0) \in Y$.) The first term of J ensures that $\underline{M}$ must be regular, and the second term tries to force $\underline{m} = \underline{f}$. If a minimiser of J exists it is then a regular solution of $(*)$, but with initial data $\underline{m}$ rather than $\underline{f}$.

Fursikov showed that given any initial condition $\underline{m}$ there exists a sequence of $\underline{f_k}$ then converge to $\underline{m}$ such that $(*)$ has a regular solution with initial data $\underline{f_k}$.

This result strongly suggests that *the set of initial data u_0 for which regular solutions exist is dense.* Indeed, suppose that $f_k(0)$ as the moments of the Dirac delta measure supported at u_0. Then the moment problem can be solved, yielding regular moments, for initial moments arbitrary close to those of δ_{u_0}. If convergence of moments implies convergence of measures then

this should imply density of regular initial data. We visited Prof. Fursikov in Moscow in 2011 to begin a collaboration on these problems.

A first step to interpreting Fursikov's result is to understand the meaning of the convergence in moments. We have developed a number of results in this direction. Essentially, if the moments correspond to (positive or signed) measures of bounded support then convergence of moments implies weak convergence of measures.

However, note that this convergence result required the measure to be bounded. In any finite-dimensional space, any sequence of moments corresponds to some measure (we are currently working on a proof of a similar result in an infinite-dimensional Hilbert space), but this may be a signed measure with unbounded support, i.e. the difference of two positive measures. If the meaure is positive then the support is known to be bounded. Conditions are known, in finite and infinite-dimensional Hilbert spaces, that ensure that a sequence of measures corresponds to a *positive* measure, which is what we require.

However, it is not clear that these conditions are satisfied for the approximating initial conditions f_k that the minimisation approach yields. In other words, while our target initial moments f come from a positive measure by construction, the initial condition f_k for which we can guarantee the existence of regular solutions of $(*)$ may not.

In fact we can show that they cannot in a much simpler example, namely the scalar (ODE). One can apply the same minimisation scheme in this case: starting with an initial positive measure δ_{x_0}, find its moments, and try to solve the moment equations with this initial data using the minimisation approach. This yields a sequence of initial moments f_k which converge to the moments of δ_{x_0}, and for which regular solutions exist. But we know that if the moments correspond to bounded positive measures then their supports converge, and it particular this would imply that arbitrarily close to x_0 there is a smooth solution that exists for all time. However, solutions of (ODE) blow up in a finite time if $x_0 > 1$, so the approximating moments *cannot* correspond to positive measures.

The problem of applying the method if one restricts to moments corresponding to positive measures is the lack of Euler–Largange equations (ELE) for the minimisation problem, a key point in Fursikov's argument to obtain the convergence of the moments. One can still try to avoid this problem by constructing a sequence of approximating problems for which the ELE holds (as in Fursikov's method). However, on the real space this procedure requires us to abandon the setting of positive measures.

This lead us to attack the problem on complex Hilbert spaces restricted to positive measures. As a first step, we have started by considering the problem for the ODE. A positive solution for the problem should throw light upon the way to an extension to the case of 3DNSE. So far, we have been able to determine a functional setting for the problem which allows the solutions in the approximated problems to become larger and larger as we approximate the limit problem. We expect to be able to take advantage of the fact of being in a complex space (where for (ODE) we *do* have density of regularity in the space of initial data) to be able to construct the Euler–Lagrange Equations that lead us to the convergence of the moments of the approximated problems and so to the convergence of the corresponding measures and their support, providing density of regularity. We hope to obtain first results in this direction in the next visit of Prof. Fursikov on January 2013 for five weeks.

Preliminay results have been presented in the international conference EQUADIFF 2011, Loughborough (UK), in the regular Seminar of the Dept. Matemática Aplicada of the Universidad Complutense de Madrid (Spain), and in a special session in the 9th AIMS Conference in Orlando, USA.