Final Publishable Summary Report

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1 Work carried out to achieve the project's objectives

My work in the project DECONSTRUCT at INRIA begun with a collaboration with the hosting group GALAAD. Thanks to the experience of the group and my knowleges in the algebraic geometry concerning tensor decomposition, we were almost immediately able to write **the first algorithm for the computation of the partially symmetric rank of all partially symmetric tensors over the complex numbers**. This solved completely the part of the main objective of my project regarding algorithms for the computation of the rank of partially symmetric tensors.

I presented this result during the Spring Semester 2011 "Algebraic geometry with a view towards applications" at Mittag-Leffler Institut where I was invited as visitor by A. Dickenstein, S. Di Rocco, R. Piene, K. Ranestad and B. Sturmfels. I started a new collaboration with K. Ranestad on a new concept of rank, and a collaboration with J. Hauenstein to write **the first effective numerical algorithm for computing the rank and the border rank of real and complex symmetric tensors** (this is still a work in progress). The interest showed by the European and the International communities on the topics of ranks and border ranks of structured tensors, gave a serious motivation to pay more and more attention to the main topic of my project, namely classify secant varieties of varieties parameterizing partially symmetric tensors and skew-symmetric tensors in terms of dimension and of rank.

Important results that I obtained on the main topic of this project are related to the **computation of the dimensions of secant varieties of Segre-Veronese varieties**: in collaboration with E. Ballico and M.V. Catalisano we wrote **a complete classification of the dimensions of secant varieties of Segre-Veronese varieties of two factors $\mathbb{P}^n \times \mathbb{P}^1$ embedded in bi-degree** (a, b). After the very famous work of Alexander and Hirschowitz on the dimensions of secant varieties of Veronese varieties, [**?**] is the first paper that can give a complete classification of the dimensions of secant varieties of a given class of varieties parameterizing tensors.

For what concerns the objective of the project to takle the problems over the field of **real numbers**, I started a collaboration with Prof. G. Ottaviani to compute the typical ranks of real ternary and quaternary cubics. This is still a work in progress since we are planning to cover the case of ternary quartics in collaboration with G. Blekhermann. I also wrote a Preprint together with E. Ballico on the relation between the complex and the real ranks of a real symmetric tensor.

Regarding my work in connections with the **applications**, beside the first quoted pqper on the decomposition of all partially symmetric tensors which is related with problems in **Signal Processing**, another work that it is worth mentioning is in collaboration with I. Carusotto. Here we use the algebraic geometry tools for the decomposition polynomials to applications to **quantum and atomic physics**.

2 The main results

–"Multihomogeneous Polynomial Decomposition using Moment Matrices" A. Bernardi, J. Brachart, P. Comon, B. Mourrain. A. Leykin editor, International Symposium of Symbolic and Algebraic Computation (ISSAC), pp. 35–42, San Jose, CA, USA, June, 2011, ACM New York.
In the paper, we address the important problem of tensor decompositions which can be seen as a generalisation of Singular Value Decomposition for matrices. We consider general multilinear and multihomogeneous tensors. We show how to reduce the problem to a truncated moment matrix problem and give a new criterion for flat extension of Quasi-Hankel matrices. We connect this criterion to the commutation characterisation of border bases. A new algorithm is described which applies for general multihomogeneous tensors, extending the approach of J.J. Sylvester on binary forms. An example illustrates the algebraic operations involved in this approach and how the decomposition can be recovered from eigenvector computation.

–"Higher secant varieties of $\mathbb{P}^n \times \mathbb{P}^1$ embedded in bi-degree (a, b)". E. Ballico, A. Bernardi, M. V. Catalisano. Communications in Algebra. 40:3822–3840 (2012).
In this paper we compute the dimension of all the higher secant varieties to the Segre-Veronese embedding of $\mathbb{P}^n \times \mathbb{P}^1$ *via the section of the sheaf* $\mathcal{O}(a, b)$ *for any* $n, a, b \in \mathbb{Z}^+$*. We relate this result to the Grassmann Defectivity of Veronese varieties and we classify all the Grassmann* $(1, s-1)$*-defective Veronese varieties.*

–"Real and complex rank for real symmetric tensors with low complex symmetric rank" E. Ballico, A. Bernardi. Preprint: `http://hal.inria.fr/hal-00693413`
We study the case of real homogeneous polynomial P *whose minimal real and complex decompositions in terms of powers of linear forms are different. In particularly we will show that, if the sum of the complex and the real ranks of* P *is smaller or equal than* $3\deg(P) - 1$*, then the difference of the two decompositions is completely determined either on a line or on a conic.*

–"Algebraic Geometry tools for the study of entanglement: an application to spin squeezed states" A. Bernardi, I. Carusotto. J. Phys. A: Math. Theor. 45 (2012) 105304 (13pp).
A short review of Algebraic Geometry tools for the decomposition of tensors and polynomials is given from the point of view of applications to quantum and atomic physics. Examples of application to assemblies of indistinguishable two-level bosonic atoms are discussed using modern formulations of the classical Sylvester's algorithm for the decomposition of homogeneous polynomials in two variables. In particular, the symmetric rank and symmetric border rank of spin squeezed states is calculated as well as their Schrödinger-cat-like decomposition as the sum of macroscopically different coherent spin states; Fock states provide an example of states for which the symmetric rank and the symmetric border rank are different.

3 Conclusions and their potential impact and use and any socio-economic impact of the project

The impact of this work will be on all the communities, pure mathematical and applied ones, that are interested in tensor decomposition, namely Signal Processing, Antenna Array Processing, Telecommunications, Phylogenetics, Chemometrics, Algebraic Statistics, Geometric complexity theory, Arithmetic complexity, Data Analysis (e.g. Independent Component Analysis qnd Quantum Physics. Our algorithm will give them an effective tool that could be use to actually compute the rank and a decomposition of a given partially symmetric tensor.