IFIOP: Inequalities on Function Spaces and Properties of Integral Operators with Applications. Final report, 10 October 2012.

This project involved the study of *integral operators*, mathematical objects devised originally to help in the solution of differential equations, but subsequently studied for their intrinsic mathematical interest. The properties we investigated include *boundedness* — roughly, by how large a factor these operators scale up or down the data that it fed into them — and the closely related ideas of *compactness* and *approximation numbers* — these tell us how accurately these rather complex objects can be described by finite tables of numbers called *matrices*, which can be processed by computers. As well as the operators, we have to consider the *function spaces* on which they act. These are, essentially, containers for the data that is fed into the operator and the transformed data that it returns; our work mostly concerns the *Lebesgue spaces*, denoted L^p. Here p denotes a number; different values of p prove to be useful in different applications. We also work with the slightly more complicated *Lorentz* spaces $L^{p,q}$ and *Sobolev* spaces $W^1_{p,s}$.

The first part of the project was devoted to two particular integral operators: Laplace type, denoted $\mathscr{L}$, and Stieltjes type, denoted $\mathscr{S}$, acting between the Lebesgue spaces $L^p(0,\infty)$ and $L^q(0,\infty)$. These two operators are closely related to each other, and to other integral transforms that play an important role in mathematical analysis and its applications. Boundedness of these operators was already understood, so we focused on compactness and on the behaviour of their approximation numbers. We obtained necessary and sufficient conditions for compactness, expressed in an explicit form and covering all $p \geq 1$ and $q > 0$.

Our study of the approximation numbers a_n of $\mathscr{L}$ and $\mathscr{S}$ was in terms of Schatten type norms $\left(\sum_{\mathbb{N}} a_n^\alpha\right)^{1/\alpha}$ and upper asymptotic estimates for the sequence $\{a_n\}_{n\in\mathbb{N}}$. First we found two-sided estimates for the Schatten-von Neumann norms of $\mathscr{L}$ in the special (Hilbert) case $p = q = 2$. This led us to the same type of estimates for the $\mathscr{S}$. In the general case, $p \geq 1$ and $q > 0$, we derived upper estimates for weighted Schatten type norms for $\mathscr{L}$. When $1 \leq p \leq q < \infty$, we obtained upper asymptotic estimates for $a_n(\mathscr{L})$.

The second part of the project concerned an operator $\mathscr{H}$, known as a *Hardy-Steklov* operator, acting between the same kind of function spaces. This is a generalisation of the well-known Hardy integral operator $f \mapsto \int_0^x f(y)\mathrm{d}y$, and is described by the formula $f \mapsto \int_{a(x)}^{b(x)} k(x,y)f(y)\mathrm{d}y$. Compared to the Hardy operator, the Hardy-Steklov operator has different properties and a wider range of applications. Some of its properties, e.g. boundedness and compactness, have been studied before by many authors. This project concentrated on applications of these earlier results to an embedding problem and on estimates for Schatten-von Neumann norms of $\mathscr{H}$ on $L^p(0,\infty)$.

The question of how to describe the Schatten-von Neumann norms of $\mathscr{H}$ was solved in the project for all $p > 1$. On the one hand, this improves some earlier upper estimates obtained in only a discrete form. On the other hand, the result provides entirely new lower estimates for that type of Schatten norm when $p > 1$ and $\alpha > 0$. Moreover, some of our auxiliary results gave conditional upper and lower estimates on approximation numbers of $\mathscr{H}$ which open the possibility of describing the asymptotic behaviour of the sequence $a_n(\mathscr{H})$.

One of the project's most interesting results connected the Hardy-Steklov operator $\mathscr{H}$ with the characterisation of a weighted norm inequality describing the embeddings of a class of weighted Sobolev spaces $W^1_{p,s}(0,\infty)$ into a Lebesgue space $L^q(0,\infty)$. The two questions are related to each other via a principle of duality which links boundedness criteria for the operator $\mathscr{H}$ on the Lebesgue spaces with necessary and sufficient conditions for the embedding inequality to hold. This duality principle helped us to characterise the inequality for $1 < p, s, q < \infty$. The remaining cases $p > 1$ and $0 < s, q < \infty$ were described by using other methods. The results obtained provide us with criteria for the validity of the embedding inequality when $1 < p \leq q$, $0 < s \leq q$ or $0 < q < p$, $p = s > 1$.

The final part of the project was devoted to the characterisation of two norm inequalities involving the Hardy integral operator H_n on a one-dimensional or two-dimensional space. In the first case, we dealt with the weighted one-dimensional operator $H_1 f(x) = \phi(x) \int_0^x f(y)\psi(y)\mathrm{d}y$ between weighted Lorentz function spaces $L^{r,s}(v)$ and $L^{p,q}(w)$ on $(0,\infty)$. The second problem concerned the weighted two-dimensional Hardy integral inequality with $H_2 f(x,y) = \int_0^x \int_0^y f(s,t)\mathrm{d}s\mathrm{d}t$. In both cases we found explicit sufficient conditions for the inequalities to hold, expressed in terms of the weights and summation parameters.

A general result obtained for H_1, from $L^{r,s}(v)$ to $L^{p,q}(w)$, provides two different sufficient conditions for the validity of the corresponding Lorentz norm inequality when $1 < s \leq r$, $1 < q \leq p$ and $1 < p \leq q$, $1 < r \leq s$. In the special case $\phi \equiv 1$, we describe the inequality by two sufficient conditions which have more compact forms and are also different depending on the relation between $1 < r < \infty$ and $0 < s < \infty$. The parameters $0 < p, q < \infty$ do not play any crucial role in this case. Finally, under some serious assumptions on the weights v and ψ, we can obtain a criterion for the Lorentz norm inequality for H_1 to hold.

As for the two-dimensional Hardy integral inequality, our condition is valid for all weights in the inequality and provides a sufficient boundedness condition for the operator H_2 from $L^p(v)$ with a weight v to $L^q(w)$ with another weight w on a quadrant in the plane when $1 < q < p < \infty$. This inequality was completely characterised by E. Saywer only in the case $1 < p \leq q < \infty$. The reverse case $q < p$ with general weights v and w remained unsolved for many years. In this project a sufficient condition for the two-dimensional Hardy integral inequality is established in the case $q < p$ and is valid for all v and w.

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