Characterization of 4-searchable series-parallel graphs

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Abstract

The set of forbidden minors for k-searchable graphs is known only when $k \leq 3$. In this paper we give the complete set of forbidden minors for 4-searchable series-parallel graphs. We give a recursive method that constructs these forbidden minors. This method can be used to construct forbidden minors for k-searchable series-parallel graphs using those for $(k-2)-$searchable graphs.

Keywords: forbidden minors, graph searching, series-parallel graphs

1 Introduction

Edge searching is an extensively studied graph theoretical problem. Its origins date back to the late 1960s in works of Parsons [37] and Breisch [11]. It was first faced by a group of spelunkers who were trying to find a person lost in a system of caves. They were interested in the minimum number of people they needed in the searching team.

There are many variants of graph searching studied in the literature, which are either application driven, i.e. motivated by problems in practice, or are inspired by foundational issues in computer science, discrete mathematics, and artificial inteligence. Network security [5] comes as the most natural application of the problem. It has strong connections with the cutwidth of a graph which arises in VLSI chip design [13] and with pebble games [27]. Because of its closeness with the layout problems, it is related to graph parameters such as pathwidth [17, 26], cutwidth [32], bandwidth [20] and topological bandwidth [31].

Assume that we want to secure a system of tunnels from a hidden intruder. We can model this system as a graph where junctions correspond to vertices and tunnels correspond to edges. We will launch a group of searchers into the system in order to catch the intruder. Let $G = (V, E)$

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denote a graph where V is its vertex set and E is its edge set. In this paper, we only consider finite connected simple graphs. We assume that every edge of G is contaminated initially and our aim is to clean the whole graph by a sequence of steps. First we place a given set of k searchers on a subset of V (allowing multiple searchers to be placed on any vertex). Each step of an edge search strategy consists of either: removing a searcher from one vertex and placing it on another vertex (a "jump"), or sliding a searcher from a vertex along an edge to an adjacent vertex.

If a searcher slides along an edge $e = uv$ from u to v, then e is cleaned if either (i) another searcher is stationed at u, or (ii) all other edges incident to u are already clean. If a searcher is stationed at a vertex v, then we say that v is *guarded*. If a path does not contain any guarded vertex, then it is called an *unguarded path*. If there is an unguarded path that contains one endpoint of a contaminated edge and one endpoint of a cleaned edge e, then e gets *recontaminated*.

An *edge search strategy* is a combination of the moves defined above so that the state of all edges being simultaneously clean is achieved, in which case we say that the graph is *cleaned*. The problem becomes cleaning the graph using the fewest searchers. The least number of searchers needed to clean the graph is the *edge search number* of the graph and is denoted $s(G)$.

If we insist that once an edge becomes clean it must be kept clean until the end of the searching strategy, then such a strategy will be called *monotonic*. It has been shown [8, 30] that forcing a search to be monotonic does not change the search number. It has been shown in [33] that finding whether a graph G can be searched by k searchers, i.e. solving the EDGE SEARCHING problem for G, is NP-complete.

One of the major problems of edge search is to characterize the graphs G such that $s(G) \leq k$ for a fixed positive interger k. A graph G is said to be *k-searchable* if $s(G) \leq k$.

Given a graph G consider the following ways of reducing it: (1) delete an edge, (2) contract an edge, (3) delete an isolated vertex. Any graph H that can be produced from G by successive application of these reductions is called a *minor* of G. We say that a graph H is a *forbidden minor* if $k < s(H)$ and if any proper minor of H has search number at most k. For fixed k, the set of forbidden minors for k-searchable graphs is called the *obstruction set*, denoted as $\mathcal{F}_k$.

It is known that edge searching is closed under the taking of minors. That is, if G contains H as a minor, then $s(H) \leq s(G)$. The theory on graph minors built by Robertson and Seymour [38] implies that the obstruction set for a minor closed family is finite. Thus, when k is fixed $\mathcal{F}_k$ is finite, since edge searching is minor closed. On the other hand no general method is known for constructing an obstruction set. Further, the size of such a set is not known either except for some initial cases. Graphs that are $1, 2$ or 3-searchable have been completely characterized previously by Megiddo et. al. [33]. Characterizing k-searchable graphs where $k \geq 4$ is left as an open problem since 1976.

The first result on forbidden minors is due to Wagner [39]. It refines Kuratowski‘s theorem [29] which says that a graph is planar (i.e. it has an embedding in the plane without any edges crossing), whenever it does not contain a subdivision of K_5 or $K_{3,3}$. Wagner showed that a graph is planar if and only if K_5 or $K_{3,3}$ are not minors of G. This gives us an exact forbidden minor characterization of planar graphs.

Many natural and immediate questions arrise after Kuratowski's theorem, which is considered as the first characterization theorem by means of minors: Can one characterize graphs that posses a certain structure? Can the characterization be in terms of forbidden minors? The answer to these question depends on the structure put on the graph [**?**]. In our paper we want the common property to be k-searchable.

As noted by Fellows and Langston in 1989 *There is no general algorithm to compute the obstruction set for a minor closed family.* Thus there exists no general method to characterize the forbidden minors.

Mainly the existing structure theory for graphs with forbidden minors has been in an approximate sense. On the other hand, it has been noted by several authors [33, 21] that it is an important research direction to continue characterizing k-searchable graphs exactly and completely when $k \geq 4$. When these characterizations are achieved, the existings approximate solutions will be improved.

There are several characterization results regarding graph minors. Let us name a few of the existing results. There are

- 35 forbidden minors for projective plane, Archdeacon, 1981.
- 110 forbidden minors for graphs with pathwidth at most 2, Kinnersley 1989, Kinnersley, Langston 1992.
- 57 forbidden minors for graphs with linearwidth at most 2, Thilikos, 2000.
- ≥ 75 forbidden minors for graphs with treewidth at most 4, Sanders, 1993
- ≥ 16629 forbidden minors for torus, Myrvold and Chambers, 2002 (which was conjectured to be more than 1000 by Glover and Huneke in 1995.)

Notice that the first three results are complete whereas the last two are partial; they give a lower bound on the number of forbidden minors, instead of defining all of them explicitly.

As far as edge searching is concerned, the only complete results are for $k \leq 3$ [33]. For $k = 4$ the only graph family for which the complete list of forbidden minors is given is the class of biconnected 4-searchable outerplanar graphs [14]. In this paper, we solve the problem of forbidden minor characterization for 4-searchable graphs partially by giving the complete list of minors for 4-searchable series-parallel graphs.

2 Preliminaries

Series-parallel graphs are themselves characterized by excluded minors by the following theorem: A graph is series-parallel if it has no minor isomorphic to K_4. A series-parallel graph has two distinguished vertices, called the *source,* s and the *sink,* t. It is defined recursively as follows. In the initial case we assume that an edge, K_2, is a series-parallel graph. Let G_1 with source s_1 and sink t_1, and G_2 with source s_2 and sink t_2 be two series-parallel graphs. We define the following graph operations: A *Series Composition* of G_1 and G_2, denoted as $G_1 \oplus G_2$, is obtained by dentifying t_1 with s_2. Thus $G_1 \oplus G_2$ has source $s = s_1$ and $t = t_2$. A *Parallel Composition* of G_1 and G_2, denoted as $G_1 \| G_2$, is obtained by identifying s_1 with s_2 and t_1 with t_2. Thus $G_1 \| G_2$ has source $s = s_1 = s_2$ and $t = t_1 = t_2$.

Let us denote the obstruction set for 4-searchable series-parallel graphs as $\mathcal{F}_{sp}$. A *base graph* is an induced subgraph of a reduced series-parallel graph that is not isomorphic to K_2 and that contains the source and the sink. Furthermore, we put the condition that a composition of 3 base graphs form a graph in $\mathcal{F}_{sp}$. Thus by using the base graphs we construct the graphs in $\mathcal{F}_{sp}$. A *base graph in n-levels* correspond to that for which the longest path from the source to the sink has length n. An examples for the forbidden minor construction for 4-searchable series-parallel graphs is given in Figure **??** using base graphs.

Let G_1, G_2 and G_3 be series parallel graphs. Consider the following operations to obtain a new graph G by taking a composition of these three graphs.

Parallel-Parallel Composition: Combine G_1 and G_2 using the parallel composition; apply the parallel composition operation to this graph and G_3. This corresponds to the parallel composition operation applied twice. Thus G is isomorphic to $(G_1 \| G_2) \| G_3$.

Parallel-Series Composition: Combine G_1 and G_2 using the series composition; apply the parallel composition operation to this graph and G_3. Hence, in a sense this is a cyclic composition that consists of first series and then parallel composition. In this case G is isomorphic to $(G_1 \oplus G_2) \| G_3$.

The base graphs that are used for constructing the graphs in $\mathcal{F}_{sp}$ by the Parallel-Parallel Composition Operation are called *Type I base graphs*. Similarly the base Graphs for constructing the graphs in $\mathcal{F}_{sp}$ by the Parallel-Series Composition Operation are called *Type II base graphs*. Thus the union of Type I and Type II forbidden minors is exactly the set of 4-searchable forbidden minors.

We use the symbol E_i, $i \geq 1$, to denote the series-parallel graph with two nodes and i parallel edges between them. The symbols $s(G)$ and $t(G)$ denote the source and the sink vertices, respectively, of a series-parallel graph G. For a series-parallel graph G we write $\overline{G}$ to denote the series-parallel graph obtained by treating $s(G)$ as a sink vertex and $t(G)$ as a source vertex.

We use $\mathtt{s}(G)$ to denote the search number of a graph. In this paper, without loss of generality, only monotone search strategies are considered (no recontamination).

A search strategy for a graph G is *connected* if, at any point of the strategy, the subgraph of G consisting of cleared edges is connected. If G is a series-parallel graph, then a search strategy for G is *semi-connected* if it is either connected, or at any point of the strategy the cleared edges induce two connected components one of which contains $s(G)$ and the other contains $t(G)$.

3 Introduction of building blocks

Let us first give the following lemma which shows that connectivity condition does not increase the number of searchers used when considering graphs with search number at most 4.

Lemma 3.1 *If G is a reduced biconnected graph with $\mathtt{s}(G) \leq 4$, then there exists a connected $\mathtt{s}(G)$-search strategy for G. Furthermore, each search strategy that uses at most $\mathtt{s}(G)$ searchers is connected.*

Proof: Let G be a reduced biconnected graph with $\mathtt{s}(G) \leq 4$. Assume that S is a search strategy that uses $\mathtt{s}(G)$ searchers and assume that S is a non-connected search. Let t be the last step that the subgraph induced by the clean edges (according to S) are connected. Call this connected subgraph G_1. Furthermore, assume that it is not possible to extend G_1. (Otherwise we continue extending the connected subgraph induced by the clean edges and take t as the last step corresponding to this.)

At step $(t+1)$ there exists an edge $e \in E$ and $e \notin E[G_1]$ such that e is cleaned and $G_1 \cup \{e\}$ is disconnected. Observe that a single searcher located at a vertex $v \in V[G_1]$ can not guard G_1 (from recontamination) since G is biconnected (otherwise v is a cut vertex). Thus at least two searchers guard G_1. Since $G_1 \cup \{e\}$ is disconnected (and it is not possible to extend G_1), after step $(t+1)$ there must be a searcher located at each end vertex of $e = uv$ where $u, v \notin V[G_1]$. After this step, however, each searcher is stuck, since S uses at most 4 searchers. Thus G can never become clean according to S, which contradicts to our original assumption. Hence S can not be a non-connected search strategy. □

Next, we introduce a notation for several classes of graphs that will be used as building blocks in our constructions. Each class we define is a set of graphs that are forbidden minors satisfying some specified properties. Those properties provide restrictions on search strategies that can be used on those graphs and on the structure of the graphs.

Let G be a graph, let v be its arbitrary vertex and let $\mathcal{S}$ be a search strategy for G. If $\mathcal{S}$ places a searcher on v prior to clearing any edge of G, then we say that v is *preoccupied* in $\mathcal{S}$. If $\mathcal{S}$ has the property that once v is reached by a searcher, v is occupied by a searcher till the end of $\mathcal{S}$, then v is called a *trap* for $\mathcal{S}$. Note that v can be both preoccupied and a trap in $\mathcal{S}$.

Thorough the paper we use two special symbols $\mathbb{s}$ and $\mathbb{t}$. Informally, they 'represent' the source and sink nodes of series-parallel graphs.

Let P and T be subsets of $\{\mathbb{s}, \mathbb{t}\}$ such that $P \neq \emptyset$, and let $k \geq 1$ be an integer. We say that $\mathcal{S}$ is a (k, P, T)-search strategy for a graph G if $s(G)$ (respectively $t(G)$) is preoccupied in $\mathcal{S}$ if and only if $\mathbb{s} \in P$ (respectively, $\mathbb{t} \in P$), and $s(G)$ (respectively $t(G)$) is a trap for $\mathcal{S}$ if and only if $\mathbb{s} \in T$ (respectively, $\mathbb{t} \in T$). In other words, the symbols in P and T indicate whether source and sink nodes are preoccupied and traps, respectively.

We now define some families of series-parallel graphs. We define $\mathcal{M}(k, P, T)$ to be the set of series-parallel graphs such that each $G \in \mathcal{M}(k, P, T)$ satisfies:

- there exists no (k, P, T)-search strategy for G, and
- there exists a (k, P, T)-search strategy for each proper minor of G.

Let $\overline{\mathcal{M}}(k, P, T)$ be defined in such a way that G belongs to this class if and only if $\overline{G} \in \mathcal{M}(k, P, T)$ If X and Y are two sets of series-parallel graphs, then

$$X \oplus Y = \left\{ G_1 \oplus G_2 \,\middle|\, G_1 \in X \wedge G_2 \in Y \right\},$$

$$X \parallel Y = \left\{ G_1 \parallel G_2 \,\middle|\, G_1 \in X \wedge G_2 \in Y \right\}.$$

4 Properties of search strategies

In this section we provide some structural properties of search strategies for selected series-parallel graphs. More precisely, our goal is to narrow down the number of possible search strategies when constructing forbidden minors in the subsequent sections.

Suppose that a series-parallel graph G is a parallel composition, $G = G_1 \parallel G_2 \parallel \cdots \parallel G_l$, where none of G_i's, $i \in \{1, \ldots, l\}$, is a parallel composition. We say that a search strategy $\mathcal{S}$ for G clears G *sequentially* if for each $i \in \{1, \ldots, l\}$ there exist integers $j_i \leq j'_i$ such that the j-th move of $\mathcal{S}$ clears an edge of G_i if and only if $j \in \{j_i, \ldots, j'_i\}$.

Lemma 4.1 *Let $G = G_1 \oplus G_2 \in \mathcal{M}(k, P, T)$, $k \leq 4$, $P, T \subseteq \{\mathbb{s}, \mathbb{t}\}$, $\mathbb{s} \in P$, $\mathbb{s} \in T$, and let $\mathcal{S}$ be any partial (k, P, T)-search strategy for G which does not clear G_i for each $i \in \{1, 2\}$. Moreover, G_1 is not a series composition of two graphs. Then, the common vertex of G_1 and G_2 is a trap for $\mathcal{S}$.*

Proof: Suppose for a contradiction that v becomes guarded in some move i of $\mathcal{S}$ is not guarded in a move $j > i$. This means, due to monotonicity, that all edges incident to v are clear. Since, by assumption, neither G_1 nor G_2 is clear at the end of move j, at least two vertices in $V(G_i) \setminus \{v\}$ are guarded, $i \in \{1, 2\}$. Since $s(G)$ is a trap and $k \leq 4$, $s(G)$ and some vertex u are the ones in $V(G_i) \setminus \{v\}$ that are guarded. But then, u is a cut vertex in G_1, which implies that $G_1 = G_1' \oplus G_1''$, where $s(G_1'') = t(G_1') = u$ — a contradiction with the choice of G_1. □

Note that the above lemma says that, if the source vertex of G is both preoccupied and a trap, then there exists a choice of G_1 and G_2 such that $G = G_1 \oplus G_2$ and the condition in the lemma holds for the common vertex of the two selected subgraphs. Thus, this lemma can be applied, e.g., for $G \in \overline{\mathcal{M}}(3, \{\mathbb{t}\}, \{\mathbb{t}\})$. In general, we obtain the following corollary in case when the sink node of G is both a preoccupied vertex and a trap.

Corollary 4.1 *Let $G = G_1 \oplus G_2 \in \mathcal{M}(k, P, T)$, $k \leq 4$, $P, T \subseteq \{\mathbb{s}, \mathbb{t}\}$, $\mathbb{t} \in P$, $\mathbb{t} \in T$, and let $\mathcal{S}$ be any partial (k, P, T)-search strategy for G which does not clear G_i for each $i \in \{1, 2\}$. Moreover, G_2 is not a series composition of two graphs. Then, the common vertex of G_1 and G_2 is a trap for $\mathcal{S}$.*

We say that a search strategy for a series parallel-graph G is *simple* if there exist graphs G_1, G_2 such that $G = G_1 \parallel G_2$ or $G = G_1 \oplus G_2$ and two integers j, j', $j < j'$, such that each clearing move i of $\mathcal{S}$, where $i \leq j$ or $i > j'$, clears an edge of G_t, $t \in \{1, 2\}$, and each clearing move $i \in \{j+1, \ldots, j'\}$ of $\mathcal{S}$ clears an edge of $G_{t'}$, $t' \in \{1, 2\} \setminus \{t\}$. Informally, $\mathcal{S}$ starts by clearing one of the subgraphs G_1, G_2, clearing it partially or completely, then the other subgraph is cleared completely, and finally, if the first subgraph has not been cleared completely till date, then the strategy finishes by clearing it.

Lemma 4.2 *Let $P, T \subseteq \{\mathbb{s}, \mathbb{t}\}$, $P \neq \emptyset$ and G be a series-parallel graph. If there exists a $(3, P, T)$-search strategy for G, then there exists a simple $(3, P, T)$-search strategy for G.*

Proof: Suppose first that G is a parallel composition, $G = G_1 \parallel \cdots \parallel G_l$, where G_i is not a parallel composition for each $i \in \{1, \ldots, \}$. Let $\mathcal{S}$ be a $(3, P, T)$-search strategy $\mathcal{S}$ for G. Assume without loss of generality that for each $i \in \{1, \ldots, l-1\}$, the event of clearing the first edge of G_i proceeds of the event of clearing the first edge of G_{i+1}. Also, among all such search strategies that 'order' the subgraphs $G_1, \ldots, G_l$ in this way take $\mathcal{S}$ to be one that starts clearing G_l as late as possible. Let j be the first move of $\mathcal{S}$ that clears an edge of G_l. If $G_1 \parallel \cdots \parallel G_{l-1}$ is clear at the beginning of move j, then the lemma follows. Otherwise, two vertices in $V(G_1 \parallel \cdots \parallel G_{l-1})$ and a vertex in $V(G_l) \setminus \{s(G), t(G)\}$ are guarded at the end of move j. Suppose for a contradiction that the next move, say move j', that clears an edge of $G_1 \parallel \cdots \parallel G_{l-1}$ occurs prior to clearing G_l completely. Suppose that move j' slides a searcher from u to v in G. If u has only one contaminated edge incident to it at the beginning of move j, then this implies that this move can be performed prior to the move j — a contradiction. Otherwise, $u \in \{s(G), t(G)\}$ and the move j clears an edge e' of G_l incident to u. Moreover, since G_2 is not isomorphic to E_1, there is exactly one contaminated edge incident to u at the beginning of move j. Thus, the order of clearing $\{u, v\}$ and e' can be changed, providing a contradiction again. Considering how $G_1 \parallel \cdots \parallel G_{l-1}$ and G_l are cleared by $\mathcal{S}$, we obtain that $\mathcal{S}$ is simple as required. $\square$

5 The Graph Classes for Building Frobidden minors

In this section we construct several classes $\mathcal{M}(k, P, T)$ for $k \in \{2, 3\}$ and $P, T \subseteq \{\mathbb{s}, \mathbb{t}\}$. The cases when $k = 2$ are straightforward and we list them in the next lemma:

Lemma 5.1 *$\mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) = \{E_1\}$, $\mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\}) = \{E_2\}$, $\mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \emptyset) = \{E_2\}$, $\mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) = \{E_2\}$, $\mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{t}\}) = \{E_3\}$, $\mathcal{M}(2, \{\mathbb{s}\}, \emptyset) = \{E_3\}$ and $\mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}\}) = \{E_3, E_1 \oplus E_2, E_2 \oplus E_1\}$.*
$\square$

Our constructions for $k = 3$ are done in two steps. In the first step, we provide a superset of a given family of graphs $\mathcal{M}(k, P, T)$. This is done in Lemmas 5.2, 5.3, 5.4, 5.5 and 5.6. Then, $\mathcal{M}(k, P, T)$ is obtained by eliminating those graphs from the supersets that are not minors. The efficiency of this method greatly depends on obtaining a relatively small superset in the first step.

The following lemma provides a superset of the family $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. This family, in its construction, utilizes the sets $\mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$, $\mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\})$, $\overline{\mathcal{M}}(2, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ and $\mathcal{M}(2, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. The latter can be obtained from $\mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$.

Lemma 5.2 *$\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \subseteq \{E_1\} \oplus \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\}) \cup \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\}) \oplus \{E_1\} \cup \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \oplus \mathcal{M}(2, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$.*

Proof: Observe that if $G \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$, then $G = G_1 \oplus G_2$. This follows from the fact that otherwise, i.e., if $G = G_1 \parallel G_2$, then $G_i \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$, which contradicts that G is a minor. Select G_1 and G_2 so that G_1 is not a series composition. Due to Lemma 4.1, it is not possible that both G_1 and G_2 are cleared partially at any point of a search strategy. Thus, we need to consider three cases.

Case 1: G_1 becomes clear and the strategy is unable to clear G_2. Then, $G_1 \simeq E_1$ and $G_2 \in \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$.

Case 2: G_2 becomes clear and the strategy is unable to clear G_1. By symmetry, $G_2 \simeq E_2$ and $G_1 \in \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\})$.

Case 3: neither G_1 nor G_2 becomes clear. By Lemma 4.1, if the common vertex v of G_1 and G_2 becomes guarded at some point, then it remains guarded in the remaining moves of the strategy. Thus, $G_1 \in \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ and $G_2 \in \mathcal{M}(2, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$.

Thus, $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$ is contained in the union of $\{E_1\} \times \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$ (Case 1), $\mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\}) \times \{E_1\}$ (Case 2) and $\mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \times \mathcal{M}(2, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$ (Case 3), which completes the proof. $\square$

We now construct a superset for the family $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$. To that end we will use in the proof of the following lemma the sets $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$, $\mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{t}\})$ and $\mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}\})$.

Lemma 5.3 *$\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \subseteq \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \cup \{E_1\} \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \cup \{E_1\} \oplus \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{t}\}) \cup \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}\}) \oplus \{E_1\}$.*

Proof: Suppose first that $G = G_1 \parallel G_2 \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$. If no partial $(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$-search strategy can clear any of the graphs G_1, G_2, then $G_i \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ for each $i \in \{1, 2\}$ and hence G is not a minor. Thus, suppose that one of the graphs G_1, G_2, say G_1, can be cleared by such a search strategy $\mathcal{S}$. Since both s and t are traps for $\mathcal{S}$, we obtain that $G_2 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. Then, if $G_2 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$ can also be cleared by some partial $(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$-search strategy for G, then G_1 also belongs to $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. Otherwise $G_1 \simeq E_1$. Thus,

$$G \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \ \cup \ \{E_1\} \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}).$$

Suppose now that $G = G_1 \oplus G_2 \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ and denote $v = t(G_1) = s(G_1)$. Select without loss of generality G_1 and G_2 in such a way that G_1 is not a series composition. By Lemma 4.1, if v becomes guarded in some move of a $(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$-strategy $\mathcal{S}$ for G, then v remains guarded in the following moves. If $\mathcal{S}$ clears G_1, then $G_1 \simeq E_1$ and $G_2 \in \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{t}\})$. If $\mathcal{S}$ clears G_2 (after clearing G_1 partially), then $G_1 \in \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}\})$ and $G_2 \simeq E_1$. If no partial $(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$-search strategy for G can clear G_1 or G_2, then $G_1 \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ and $G_2 \in \mathcal{M}(3, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. This, however, implies that G is not a minor. Thus, if G is a series composition, then we obtain that

$$G \in \{E_1\} \oplus \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{t}\}) \ \cup \ \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}\}) \oplus \{E_1\}.$$

□

In the next lemma, in order to construct the family of graphs $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$, we will use the families $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$, $\mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \emptyset)$, $\mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{t}\})$ and $\overline{\mathcal{M}}(2, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$. Note that the latter 'provides' $\mathcal{M}(2, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$.

Lemma 5.4 $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\}) \subseteq \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \cup \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \emptyset) \oplus \{E_1\} \cup \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{t}\}) \oplus \mathcal{M}(2, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$.

Proof: To find $G \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$ suppose first that $G = G_1 \parallel G_2$. Let $\mathcal{S}$ be a $(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$-search strategy for G. Due to Lemma 4.2, we consider only simple strategies. Thus, by symmetry, we obtain $G_1, G_2 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$ and hence

$$G \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}).$$

Suppose now that $G = G_1 \oplus G_2$ and denote $v = t(G_1) = s(G_2)$. Select without loss of generality G_1 and G_2 in such a way that G_1 is not a series composition. By Corollary 4.1, if v becomes guarded in some move of a $(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$-strategy $\mathcal{S}$ for G, then v remains guarded in the following moves. This in particular implies that at no point of $\mathcal{S}$ both G_1 and G_2 are cleared partially. If G_1 becomes clear at some point of a search strategy, then $G_2 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$. Thus, G is not a minor. Hence, suppose that G_1 does not become clear, which in particular implies that at least one searcher is present on a vertex in $V(G_1) \setminus \{v\}$.

If G_2 becomes clear at some point of a search strategy, then (since t is a trap) $G_1 \in \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \emptyset)$ and $G_2 \simeq E_1$.

Finally, suppose that none of the graphs G_1, G_2 becomes clear in a search strategy. Thus, $G_1 \in \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{t}\})$ and $G_2 \in \mathcal{M}(2, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. Therefore, if G is a series composition, then

$$G \in \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \emptyset) \oplus \{E_1\} \ \cup \ \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{t}\}) \oplus \mathcal{M}(2, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}).$$

$\square$

Note that $G \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$ if and only if $\overline{G} \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\})$. Thus, Lemma 5.4 immediately gives the following.

Lemma 5.5 $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\}) \subseteq \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \cup \{E_1\} \oplus \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \emptyset) \cup \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \oplus \mathcal{M}(2, \{\mathbb{t}\}, \{\mathbb{s}\})$. $\square$

Lemma 5.6 $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}\}) = (\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\}) \cup \{E_1\}) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \cup \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \cup \{E_1\} \oplus \mathcal{M}(2, \{\mathbb{s}\}, \emptyset) \cup \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{t}\}) \oplus \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}\})$.

Proof: Let first $G = G_1 \parallel G_2$, where $G \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}\})$. Due to Lemma 4.2, we consider only a simple $(3, \{\mathbb{s}\}, \{\mathbb{s}\})$-search strategy $\mathcal{S}$ for G. If one of the graphs G_1 or G_2, say G_1, becomes clear prior to clearing any edges of G_2, then $G_2 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\})$. Thus, if a particular G_2 has a $(3, \{\mathbb{s}\}, \{\mathbb{s}\})$-search strategy that at some point guards only s and t, then $G_1 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. Otherwise, $G_1 \simeq E_1$. If none of G_1 or G_2 becomes clear in $\mathcal{S}$, then, by symmetry, we obtain that $G_i \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$, $i \in \{1, 2\}$. Thus, if G is a parallel composition, then we obtain

$$G \in \big(\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\}) \cup \{E_1\}\big) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \ \cup \ \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}).$$

Suppose now that $G = G_1 \oplus G_2$. Select without loss of generality G_1 and G_2 in such a way that G_1 is not a series composition. By Lemma 4.2 we need to consider only simple search strategies $\mathcal{S}$.

If $\mathcal{S}$ clears G_1 first and is unable to clear G_2, then $G_1 \simeq E_1$ and $G_2 \in \mathcal{M}(2, \{\mathbb{s}\}, \emptyset)$.

Suppose now that $\mathcal{S}$ first clears G_1 partially and then $\mathcal{S}$ is unable to clear G_2. By Lemma 4.1, $s(G_2)$ is a trap for $\mathcal{S}$ and hence we have $G_2 \in \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}\})$. Since each such G_2 is not in $\mathcal{M}(2, \{\mathbb{s}\}, \emptyset)$, we obtain that G_1 cannot be cleared completely prior to clearing an edge of G_2. This implies that $G_1 \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$.

Finally note that, since t is not a trap for $\mathcal{S}$, we do not need to consider the case when $\mathcal{S}$ first clears G_1 partially and then clears G_2.

We obtain that, if G is a series composition, then

$$G \in \{E_1\} \oplus \mathcal{M}(2, \{\mathbb{s}\}, \emptyset) \ \cup \ \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \oplus \mathcal{M}(2, \{\mathbb{s}\}, \{\mathbb{s}\}),$$

which completes the proof of the lemma. $\square$

We finish this section by stating the families $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$, $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$, $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$ and $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\})$. Recall that Lemmas 5.2, 5.3, 5.4, 5.5 and 5.6 provide the supersets of those families. For each graph in a particular superset we check whether indeed there exists no 3-search strategy with appropriate constraints that clears it and whether the graph is a minor. Since this step is straightforward, we omit it and the resulting families are shown in Figure 1, where the dashed arrow pointing an edge and two parallel edges indicates that we obtain another minor by replacing the single edge with two parallel edges and vice versa. Thus, in particular, $|\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}\})| = 11$.

6 Forbidden minors of Type I

In this section we construct the family of forbidden minors of Type I, which is $\mathcal{M}(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$.

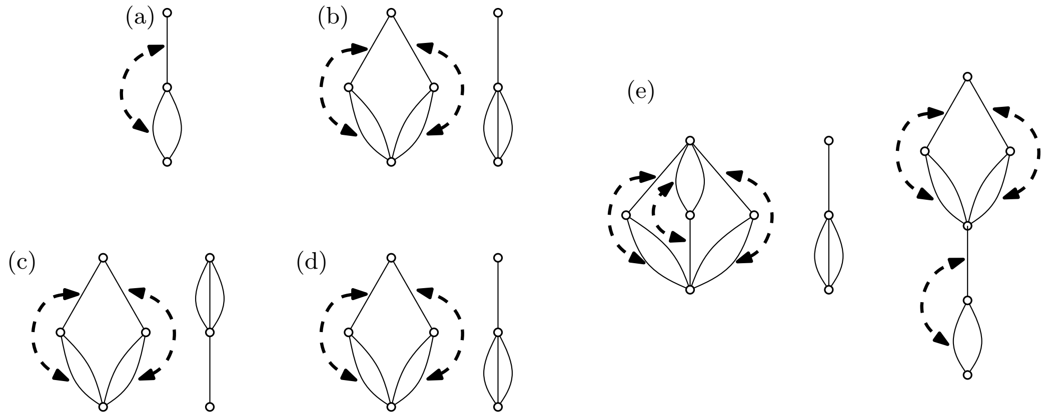

Figure 1: (a) $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$; (b) $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$; (c) $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$; (d) $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\})$; (e) $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}\})$

The proof of the following lemma exploits the families $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$, $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\})$, $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$, $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$, $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}\})$ and $\mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$ in order to construct $\mathcal{M}(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. The following lemma is crucial in the sense that it allows us to obtain our set of 4-searchable Type I forbidden minors by using the graphs in $\mathcal{M}(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$.

Lemma 6.1 $\mathcal{M}(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \subseteq \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\}) \oplus \{E_1\} \cup \{E_1\} \oplus \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\}) \cup \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \oplus \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \cup \{E_1\} \oplus \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}\}) \cup \overline{\mathcal{M}}(3, \{\mathbb{s}\}, \{\mathbb{s}\}) \oplus \{E_1\}$.

Proof: First observe that $G \in \mathcal{M}(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$ is not a parallel composition of two graphs. Denote $s = s(G)$ and $t = t(G)$. Thus, $G = G_1 \oplus G_2$. Let $\mathcal{S}$ be a partial $(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$-search strategy for G. We consider several cases depending on the behavior of $\mathcal{S}$, where in view of Lemma 4.2, we assume $\mathcal{S}$ to be simple.

Case 1: $\mathcal{S}$ first entirely clears G_1. Since s is a trap for $\mathcal{S}$, $G_2 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$ and $G_1 \simeq E_1$.

Case 2: $\mathcal{S}$ first entirely clears G_2. By symmetry, $G_2 \simeq E_1$ and $G_1 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}\})$.

Case 3: $\mathcal{S}$ first clears G_1 partially, then $\mathcal{S}$ is unable to clear neither G_1 nor G_2. Note that, after clearing G_1 partially, $t(G_1)$ is guarded and, in general, at least one vertex (namely s) in $V(G_1) \setminus \{t(G_1)\}$ is guarded. If s is the only guarded vertex in $V(G_1) \setminus \{t(G_1)\}$, then obtain that $G_1 \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ and $G_2 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. If two nodes in $V(G_1) \setminus \{t(G_1)\}$ are guarded, then $G_1 \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ and $G_2 \in \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$. In the latter case, however, each graph is $(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$-searchable.

Case 4: $\mathcal{S}$ first clears G_2 partially, then $\mathcal{S}$ is unable to clear neither G_1 nor G_2. By symmetry, $G_2 \in \mathcal{M}(3, \{\mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) = \overline{\mathcal{M}}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ and $G_1 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$ or $G_1 \in \mathcal{M}(2, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$.

Case 5: $\mathcal{S}$ first clears G_1 partially, then $\mathcal{S}$ clears G_2. We have $G_2 \simeq E_1$ and $G_1 \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}\})$.

Case 6: $\mathcal{S}$ first clears G_2 partially, then $\mathcal{S}$ clears G_1. By symmetry, $G_1 \simeq E_1$ and $G_2 \in \mathcal{M}(3, \{\mathbb{t}\}, \{\mathbb{t}\}) = \overline{\mathcal{M}}(3, \{\mathbb{s}\}, \{\mathbb{s}\})$.

□

By investigating each graph in the superset obtained in Lemma 6.1 (by checking whether the graph is a minor and whether it is not $(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$-searchable) we obtain that $\mathcal{M}(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$ is as shown in Figure 2.

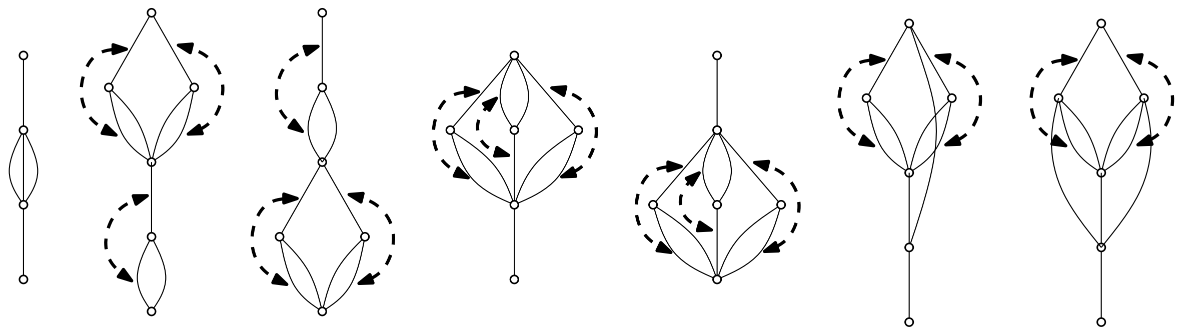

Figure 2: $\mathcal{M}(4, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$

7 Forbidden minors of Type II

In this section we provide Type II forbidden minors. This is done by constructing the class $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{t}\})$.

Lemma 7.1 $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{t}\}) \subseteq \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\}) \cup \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \cup \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\}) \parallel \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\}) \cup \{E_1\} \oplus \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\}) \cup \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\}) \oplus \{E_1\}$.

Proof: Observe that if $G \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{t}\})$, then $G = G_1 \parallel G_2$. This follows from the fact that otherwise, i.e., if $G = G_1 \oplus G_2$, then $G_i \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{t}\})$, which contradicts that G is minimal in the minor order.

First, assume that $G = G_1 \parallel G_2 \parallel G_3$. Then, it is clear that, for every $i = 1, 2, 3$, $G_i \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{s}, \mathbb{t}\})$.

Next, consider the case where $G = G_1 \parallel G_2$ where none of G_i is a parallel composition, for $i = 1, 2$. We differentiate between the cases where it is possible to clean one of G_i or not. Since we are taking the minimal graphs in the minor order, this would correspond to taking that graph isomorphic to E_1.

To start with, assume that none of G_i is an edge. Since we can assume that every search is a simple search, we know that initially one of G_1 or G_2, say G_1, is cleaned partially. Then both of G_1 and G_2 are the graphs that are series composition in the sets $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ or in $\mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$.

Otherwise, assume without loss of generality that G_1 is an edge. Let $G_2 = G_3 \oplus G_4$. Then by Corollary 4.1 and by a similar discussion given in Lemma 5.4, either $G_3 \in \mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{s}, \mathbb{t}\})$ and G_4 is isomorphic to E_1; or G_3 is isomorphic to E_1 and $G_4 \in \mathcal{M}(3, \{\mathbb{s}, \mathbb{t}\}, \{\mathbb{t}\})$. This completes the construction of a superset of $\mathcal{M}(3, \{\mathbb{s}\}, \{\mathbb{t}\})$.

□

8 Conclusion

In this paper we give the complete list of graphs that are forbidden minors for 4-searchable series-parallel graphs. This is a partial result on the construction of the forbidden minors for 4-searchable graphs. A natural approach is to start with completing this construction and continue with generalizing it to forbidden minor construction for k-searchable graphs for $k > 4$.

Our main future objective on this problem is to solve the forbidden minor characterization problem for k-searchable graphs when $k = 4$ and use this construction technique in order to generalize the result to any $k \geq 5$.

Once all of the forbidden minors are found, characterizing the 4-searchable graphs becomes possible. We have to make a clear definition of those graphs that do not contain any of the forbidden graphs we have found as minors.

Firstly making this definition for biconnected components will be the common strategy [33, 45]. Finally, we generalize the definition to general graphs by explaining how these biconnected components can be combined to form a chain of biconnected components attached at cut vertices. The cut vertices are those whose removal makes the graph disconnected.

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