

Summary Report for Project “Complexity of CSPs”

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This is a summary of the results of the research project “Complexity of CSPs”. For background and further details on each topic I refer to my research proposal. Following this proposal, my main aim has been the study of the efficient approximability of counting problems in constraint satisfaction (#CSP). A second line of research was related to the investigation of resource bounded resolution. After the first few months of work on this topic, however, I saw a need to adapt my work plan. New publications – to be discussed below – implied tremendous progress on several problems related to the second part of my research proposal. This progress made by other research groups necessitated a stronger focus on that second part, as further efforts by these other groups became more likely. After the first 3 months of the project, I therefore focused exclusively on the topics “#BIS” and “Homomorphism Functions” that can be found in my working plan under II.1 and II.2.

For convenience I provide here a short list containing the objectives met so far. Further details will be given below. The numbers indicate milestones from my proposal.

I.1 Resolution: Ruled out the applicability of several lower bound techniques.

II.1 #BIS: Determined implications of the connection between #BIS and linear Datalog.

II.2 Homomorphism Functions: Developed new algorithms, published in [SST12, Thu12].

1 Resource Bounded Resolution

Following the work plan, I started working on the problem of resource bounded resolution. I investigated techniques for establishing lower bounds for resolution refutations of small width. Within the time devoted on this project it turned out that the existing techniques examined, especially the approach of [Hak85], could not be extended to yield strong results in our context.

2 #BIS

This topic considers the problem of approximating the number of so-called *independent sets in bipartite graphs* (#BIS, for short) which is of central importance to the field of approximate counting. In particular, this problem is neither known to be efficiently solvable, nor is there particularly strong evidence that it is not efficiently solvable. Our work carried out on this topic falls in two categories, a structural one, and a directly algorithmic one.

Work on Structural Aspects of #BIS

My study of structural aspects of #BIS has been paralleled by research training on the *logical* and *algebraic* approaches to CSP, as described in my project proposal. This work has been carried out together with Victor Dalmau (UPF, Barcelona) and Andrei Bulatov (Simon Fraser University).

First of all, it is important to note that the solution space of #BIS can be characterized as a so-called *distributive lattice* – a structure intensively studied in combinatorics and algebra. This structure can be captured algebraically using so-called *distributive lattice polymorphisms* which have been applied before in the algebraic approach to the CSP. On the other hand, #BIS can be described by linear Datalog. Our aim was to establish a formal connection between the two. Although we only achieved partial results here, the implications of these two characterizations have become clear. Using these we can develop a more general description of the approximate counting problems which are of computational difficulty comparable to #BIS. Further algorithmic consequences, however, seem unlikely. We plan to publish these findings.

Algorithmic Aspects

In a second direction, together with my supervisor Albert Atserias and Elitza Maneva (Universitat de Barcelona), we considered direct algorithmic approaches to the $\#BIS$ problem. This has also been part of my approach to extend my knowledge of algorithmic approximation techniques. A first approach incorporated the study of new Markov Chain Monte Carlo algorithms. While for several such algorithms it has been known before that they do not yield efficient solutions to this problem [MWW09], we developed new algorithms specifically aiming to overcome these issues. Despite interesting partial results, however, correspondence with a research group in England provided us with a proof that our approach does not yield efficient solutions either.

3 Homomorphism Functions

Work on this topic started as a continuation of a project I had carried out during my time in Berkeley in 2010. There, together with Alistair Sinclair and Piyush Srivastava we investigated the approximability of the so-called *antiferromagnetic Ising model*. The most important quantity describing such models – their partition function – is a special homomorphism function of the form $\text{hom}(\cdot, H)$ for a weighted graph on only 2 vertices. We developed an algorithm for many of these models on degree-bounded graphs [SST12]. The case for unbounded degrees has been shown shortly afterwards by another group of researchers [LLY12]. We also worked on providing evidence that models not solvable by our algorithm are NP-hard. This has recently been shown by a different group [SS12].

This concurrence situation motivated me to focus more directly on just this single topic. Together with Albert Atserias and Elitza Maneva we tried to develop algorithms for more general models known as Potts models. Apart from partial results however, we could not achieve significantly new breakthroughs.

4 Approximating $\#k$ -SAT

Work and training on approximation techniques performed on my project have revealed a simple method to develop a new algorithm for the so-called $\#k$ -SAT problem. This problem is related to homomorphism functions and is more generally a special counting constraint satisfaction problem ($\#CSP$). The new algorithm for this problem is published in [Thu12] and solves the problem of approximating the $\#k$ -SAT problem. It is currently the fastest algorithm for this important problem.

References

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