Growing rods

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Filamentary structures can be observed in nature at all scales, from the microscopic chains of molecules to the macroscopic braided magnetic flux tubes in solar flares. Due to their geometric similarity and despite their widely different length scales and microscopic structures, filaments of all sizes seem to grow, move, and change shape according to universal laws. For instance, when a straight rope is twisted sufficiently, it will begin to coil on itself. The same change of configuration is observed to occur in bacterial fibers, DNA molecules and telephone cables. Understanding the growth, formation and dynamics of these fundamental structures is not only of intrinsic theoretical interest, but it also lies at the heart of a host of important processes in biology, physics, and engineering.

The foremost objective of this research project has been the developments of a unified mathematical framework allowing to study the fascinating interplay between the mechanics and the growth of various biofilaments as observed in bacterial fibers, bacterial filaments, fungi, stems, roots, tendrils, neurons, umbilical cords, tendons, arteries, and the spine, to name but a few.

Our work resulted in three core publications containing major theoretical contributions with far reaching practical applications. In [1], we proposed a general theory for the growth of a filament of arbitrary material properties. In particular, we emphasised cases of incompatible growth. This phenomenon is at the heart of the diversity of shapes and functions observable in both the plant and animal kingdoms. It occurs when a growing body attempts to expand in a way that is incompatible with the geometric shape of its surrounding. It results in structures the parts of which are stressed even when no external loads are applied. The parts are thereby driven to non-linear regimes and consequently, the whole structure acquires emergent mechanical properties often very different from that of its constituents. As illustrative applications of the theory, [1] includes fully detailed studies of a growing ring and of rods attached to various rigid foundations. This work was also the theoretical backbone behind Dr. D.E. Moulton's success at proposing a mechanical theory explaining the diversity and evolution of sea shells created both by extinct and contemporary species.

Although generic aspects of the mechanic of slender bodies can be understood under the framework described above, many natural filaments are constituted of sub-filaments arranged in different types of bundles. There are to date no general theory allowing to predict the effective properties of such bundles. For the simplest case of birods, that is two elastic filaments interacting elastically, a theory specifically devised to study DNA has been developed by Maddocks (EPFL) and collaborators. In [2] we introduced an alternative theoretical framework that allows to include growth in the problem. Also we made use of natural assumptions which are pertinent to many practical applications (cf. below) but not to DNA. In that sense our results complement that of Maddocks *et al.* It is also complementary in another aspect: our framework allows to obtain the effective properties of the structure from that of the sub-filaments. Again this approach is irrelevant to the study of DNA for which the mechanical properties of individual strands of the double helix are unknown but it is extremely useful for most other biological applications as well as for technological applications. The main discovery in [2] is that the traditional balance of forces and couples acting on each section as expressed by the Kirchhoff equations for rods is not the best description of bundles. Instead, we showed how generalised stresses can be defined and how to obtain and solve the corresponding balance equations.

Finally, our most promising results are the theorems proved in [3]. Until now, the stability of mechanical equilibria of one-dimensional systems (e.g. filaments) could only be established for a few particular examples and/or numerically for each particular equilibrium. This was a major problem restricting the practical usefulness of the theories mentioned above since their intrinsic geometrical non-linearity tends to lead to a large (possibly countably infinite) number of equilibrium states. However only those states that are stable are practically observed (E.g. a pencil balancing on its sharp end is theoretically at equilibrium. But practically, it never stays up). The tools developed in [3] allow to establish the stability of (almost) all the equilibria of such systems without requiring detailed knowledge of the acquired shape i.e. general topological information is sufficient. Even finding all the stable equilibria of a planar spring in tension used to be a complicated problem. We show in [3] how our results solve it with ease. Another advantage of this approach, particularly relevant to engineering applications, is that it allows to understand how to modify a system so as to stabilise particular equilibria that would otherwise be unstable. This result transcends mechanics since the theorems apply to optimisation problems in general.

Besides the three major contributions [1, 2, 3], we also studied applications of our theories to real life systems. A collaboration with a team of biologists led by Angela Hay (Max Plank Institute, Köln) led to a multi-scale model of the seed ejection mechanism of the popping cress (*Cardamine Hirsuta*) [4]. The seed pods of this 30 cm high herb contain two coiling valves. Their motion flings the seeds up to a meter away from the parent plant. By developing a sequence of models relevant to each scale of the problem, we bridged the mechanical properties of the cellular walls themselves influenced by the expression of specific genes to the distribution of seeds around the parent plant. At the heart of the mechanism lies a system of two effective layers one of which shrinks while the other does not: i.e. a growing birod which we accurately modelled by applying the theory developed in [2, 5].

Muscular thin films are bio-hybrid millimetre scale devices constituted of an inert elastic layer on top of which a thin film of muscular tissues has been grown. In a typical setup, an external electric signal triggers a coordinated contraction of the muscular tissue. The active layer shortens and the whole structure bends. This millimetre scale device is conjectured to have a number of practical applications: microscopic tweezer, microscopic pump, actuator of various kinds. As a proof of the versatility of the technology, an artificial bio-mimetic jellyfish has been built by Kit Parker and collaborators (Harvard University). The experimental work is currently guided by cumbersome numerical simulations. During Saurav Aryan's internship at Oxford, we successfully applied the theory developed in [2, 5] to produce effective models of these morphing bilayer. The student is currently working on the text.

Although our research has been motivated by biological applications, from stents to foldable tents, slender highly deformable structures are used in a wide range of technological applications. We bring [2] and [3] together in [5] where we showed how to recover the results of [2] from a variational perspective thereby enabling the theorems developed in [3]. To illustrate the theory, we revisit the morphing ladder proposed by Lachenal et al. (Bristol) as a potential deployable arm for space engineering applications. We showed that the proposed design may assume overlooked non-linear equilibria and under which conditions these may be stable. If unexpected these may have catastrophic impact on a mission. On the other hand, when properly understood they could be triggered voluntarily as they lead to an arm with advantageous mechanical properties.

References

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