GENERALIZED HILBERT SPACE COMPRESSION AND COARSE GEOMETRY OF DATA SETS

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1. Project objectives

A key technique used to gain information on a mathematical object is to embed it into another object that one understands better. For example, people have thought for a long time that the world was flat and indeed, it is not at all trivial to prove that it isn't! That is... unless you take a space rocket and look at the earth as a subset of outer space. From there, the fact that the world is round is actually quite obvious.
A parallel reasoning led to a breakthrough in the study of certain algebraic structures. In particular, one can consider a structure that mathematicians call "a group". Such groups have direct applications in real life, but are often very difficult to understand. An algebraic object that is sometimes better understood is a "Hilbert space", or more generally an "L_p-space", where p is a parameter that can take any value greater than 1. In [6], the authors introduce the (non-equivariant) compression $\alpha(G)$ of a group G as a number between 0 and 1 that describes how much G must be "compressed" in order to fit into a Hilbert space. The name "compression" is actually not so well chosen: one would expect that a high compression value would mean that you need to compress the group a lot in order to fit it into a Hilbert space. By definition, the converse is actually true. If the compression is high, then the group fits rather well into a Hilbert space and we can use everything we know on the surrouding Hilbert space to gain information on the group G in question.
For example, if $\alpha(G) > 1/2$, then it was shown in [6] that G must satisfy the infamous Yu's property (A). Also, as long as the compression is not terribly low, i.e. as long as $\alpha(G) > 0$, people showed that G satisfies the Novikov and Baum-Connes conjecture!
In this project, we studied two distinct aspects of compression.

1.1. *Non-equivariant compression.* Sometimes, a given group embeds very poorly into Hilbert spaces. In that case, we study embeddings into other spaces that we know a little bit less about, but that we still understand rather well. More concretely, we study embeddings into L_p-space and the corresponding L_p-compression. Precisely, we investigate what exactly are the relationships with Yu's property (A), what techniques can be used to calculate the L_p-compression of a given group and how a further refinement of compression, called the compression gap, can increase our understanding of fascinating groups.

1.2. *Equivariant compression.* An important feature of a group is that it can "act" in various ways on all kinds of sets. Knowing that a group acts e.g. "properly isometrically" on say "a Banach space", implies that your group satisfies a variety of interesting conditions and conjectures. Combining this point of view with the non-equivariant compression view, one is led to the introduction of the "equivariant compression" of a group G, denoted $\alpha^*(G)$. Again, this is a number between 0 and 1 that can be associated to a group and that contains crucial information on the group. For example, $\alpha^*(G) > 0$ implies that G has the Haagerup property and hence satisfies the infamous Baum-Connes conjecture and related Novikov conjecture. Moreover, $\alpha^*(G) > 1/2$ implies amenability of G, showing that the equivariant compression value summarizes interesting information on the group. In this project, we study novel generalizations of the equivariant compression, such as the equivariant L_p-compressions. We investigate how the equivariant compression behaves under group constructions, such as graph products and direct limits. We also ask what can be said about the equivariant compression of the very fascinating class of locally compact hyperbolic groups.

2. Results and their potential impact

During this Marie Curie fellowship, we made crucial progress leading to a better understanding of compression, groups and group constructions. Among other things, we have proven results that describe the behaviour of non-equivariant compression under wreath products of metric spaces [4]. Other results describe the behavior of the compression gap, the equivariant compression and the asymptotic dimension under graph products [1]. This leads to a much better understanding of such products and may thus become part of the groundwork for further research on such groups. We also discovered a way to calculate the equivariant compression of a group through its subgroups. This result can be reinterpreted to describe the behavior of the equivariant compression under direct limits [3] and it led to new and improved bounds on the compression of several groups. The result gives a new tool to researchers to try to calculate the equivariant compression of the group they are studying, thus revealing new information about the group. We also discovered connections between locally compact hyperbolic groups and their equivariant L_p-compressions [5]. In particular, we show that they admit a proper affine isometic action on an L_p-space for any p sufficiently large. In addition, we show that $\alpha_p^*(G) > 1/p$ for all such p. I would like to restrict further elaboration to the following three points.

2.1. *On locally compact hyperbolic groups.* The class of locally compact hyperbolic groups has only recently been introduced: world-wide, only 2 preprints existed on the subject. Due to the Marie Curie funding, we were able to play a pioneering role in the study of these groups. We discovered that the locally compact hyperbolic groups hold the key to answering open questions in seemingly unrelated fields. For example, we used them to obtain a breakthrough in the classification of sharply n-transitive actions on compact sets [2]. There is much mathematical interest and a lot of future projects are to be expected.

2.2. *Other locally compact properties.* Our study of locally compact hyperbolic groups can be seen as a new trend in mathematics. Indeed, so far, metric properties were only studied for finitely generated groups as they can naturally be equipped as metric spaces. However, the fact that hyperbolicity, a metric property, can also be defined for locally compact groups, led to the question whether other metric properties could be defined in this locally compact setting. In fact, people started studying property (A) in a locally compact setting as well. We have contributed to this study by giving a natural definition of compression for locally compact groups and by showing that the group must have property (A) if its compression is strictly greater than $1/2$.

2.3. *On graph products.* The project on graph products had a big impact, as it implies that the Haagerup property for groups is closed under the formation of graph products. This is astonishing, as it was already known that this property is not closed under semi-direct products and amalgamated free products. The project had much mathematical interest, and many mathematicians have requested further details from us. In fact, follow up papers have already appeared. Moreover, it seems that our algebraic methods seem generalizable to a more general result by translating them to a geometric language. Our result further has direct ties with the infamous Baum-Connes conjecture.

References

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