Reliability Optimization for Series Systems under Uncertain Component Reliability in the Design Phase

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Abstract

We develop an optimization model to determine the reliability design of critical components in a system. Since the system is under a service contract, a penalty cost should be paid by the OEM when the total system down time exceeds a predetermined level, which complicates the evaluation of the life cycle costs (LCC). Furthermore, in the design phase for each critical component, all the possible designs are subject to uncertain component reliability. We proposed three evaluation methods considering different levels of uncertainty in the model, according to the numerical results, the full-uncertainty method which includes the randomness of the number of failures as well as the randomness of the failure rates performs very well. And ignore the two types of uncertainty may results in huge LCC.

Keywords: Capital goods, Reliability optimization, Performance-based contracting, Life cycle costs

1. Introduction

Capital goods are machines or products that are used by manufacturers to produce their end-products or that are used by service organizations to deliver their services. Advanced technical systems such as medical systems, manufacturing systems, defense systems are examples of capital goods that are critical for the operational processes of their customers. System downtime of these capital goods can have serious consequences (e.g., millions of euros of reduced production output, extra waiting time of passengers, failure of military missions) and maintaining these high-tech systems are too challenging for customers to take care of by themselves. As a result, the original equipment manufactures (OEMs) provide after-sale service to their customers to keep the availability of the systems on certain levels through service contracts. Different types of service contracts were mentioned in Cohen et al. [4], among which, performance-based contracts (PBC) are novel agreements promising

a predetermined level of availability to meet the customers' objectives. Therefore under a PBC, customers pay for services according to the system performance. When the system performance fails to meet the predetermined level of availability, OEMs need to pay a penalty cost to their customers.

Under a PBC, one of the OEM's major concern is the life cycle cost (LCC), which is defined as the total cost incurred in the design/development, production, operation, maintenance, support, and final disposition of a system over its anticipated useful life span (Barringer and Weber [2]). The results of Öner et al. [19] showed that the summation of maintenance cost and downtime cost is larger than the acquisition cost and constitutes a significant portion of the LCC. These service costs are incurred by system failures which are highly dependent on system designs. Therefore, reliability design decision should consider LCC.

The customers of capital goods measure the availabilities of these complex systems at the end of service contract periods. The realized availabilities of the capital goods should meet the required performance levels. According to the PBCs, when the overall downtime of a system exceeds its targeted downtime, the OEMs will pay a penalty cost to their customers for not meeting the target. Therefore, it is important for the OEMs to calculate the probability that the total downtime exceeds the targeted downtime of a given system, so that they can choose the optimal option for system reliability design considering system availability.

In reality, engineers have to select a certain design from all possible alternatives for each critical component in a system during the design phase. The outcome of any development process for a certain design is uncertain with respect to the reliability requirement. For example, since the failure mechanisms of some emerging technologies (e.g., Micro-Electro-Mechanical Systems) are complex, it is often difficult to predict the actual reliability behaviors of the critical components before the development. Therefore newly-designed devices have been found to have different component reliabilities than the expectations after the completion of design. The uncertainties in component reliabilities can lead to large deviations of the realized system availabilities from the expected system availabilities (i.e., point estimates for the system availabilities). In this case, the uncertainty in component reliabilities also needs to be considered in the decision making of system reliability design under a PBC.

In this paper, we attempt to solve a system reliability design problem by minimizing the LCC and considering both uncertain component reliability and interval availability under a PBC, which has not been studied yet. The contribution of the paper is as follows: 1) we take uncertainty in the failure rates of the critical components into consideration. The uncertainty

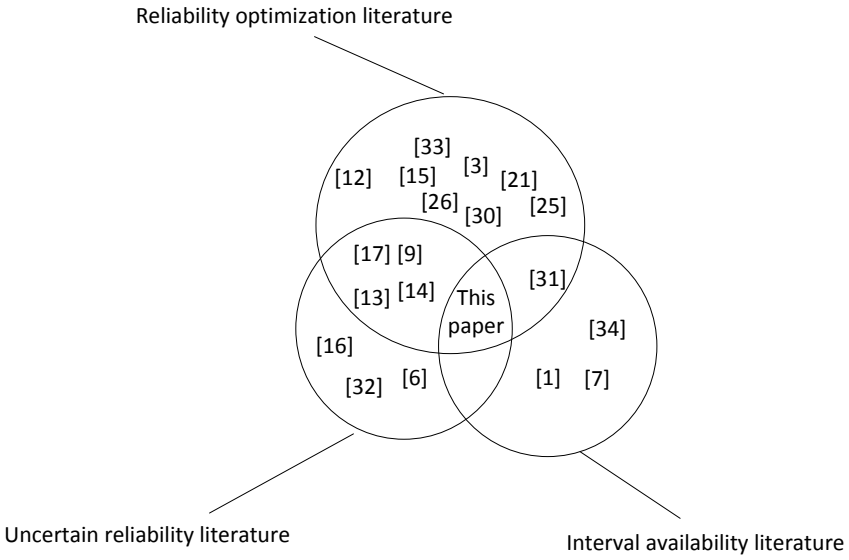

Figure 1: Literature review

in the failure rates is coming from the design phase of an advanced system; 2) in order to reduce the time to calculate the total downtime for a certain reliability design, we propose an approximate method to evaluate the total downtime of a given design in negligible time; 3) we compare results from different evaluation methods: a stochastic method ignoring the uncertain failure rate distribution, a deterministic method using expected total downtime as the actual total downtime and our own model considering both stochastic nature of the number of the system failures and the uncertain failure rates. 4) According to our numerical results, it is very important to consider the two levels uncertainty in the decision making process, ignoring the uncertainties can results huge LCCs.

The remainder of the paper is organized as follows: in Section 2, we briefly review related literature. Section 3 gives the model description and model formulation. We propose three approximation evaluation methods in Section 4. The numerical results and managerial insights of evaluation and optimization are given in Section 5. Conclusions and directions for future research are presented in Section 6.

2. Literature

According to the characteristics of our problem, we shall briefly review the literature regarding interval availability, reliability optimization and uncertain component reliability as described in Section 1.

As defined by Nakagawa and Goel [18], interval availability is the fraction of time that a system is operational during a period of time $[0, T]$. Several methods have been proposed to measure interval availability during a service period in the literature. Takacs [23] first derived the probability distribution function, the limiting distribution and the moments of interval availability in closed form. Due to the computational difficulty, De Souza E Silva and Gail [6] and Al Hanbali and van der Heijden [1] developed several methods to compute the interval availability numerically.

Regarding reliability optimization, a lot of work have been done in this area since the 1990s (Kuo and Wan [14]). For example, Mettas and Kallenberg [16] provided the minimum required reliability for each component of a system in order to achieve a system reliability goal with minimum cost, and the cost function for each component in this paper has been introduced by other papers as well, such as Huang et al. [8], Öner et al. [20] and Jin and Wang [13]. Many papers maximize the system reliability by different techniques. For example, a random search process has been proposed by Beraha and Misra [3] to assign the optimal reliability to each stage of a multi-stage system. Hwang [11] used sequential unconstrained minimization, and Li and Haimes [15] developed a 3-level decomposition approach to allocate the resource among subsystems optimally.

Some papers also built reliability allocation models to find optimal warranty policies for systems sold with traditional warranty contracts. For example, to minimize the system LCC, Monga and Zuo [17] used genetic algorithms to solve the optimization problem and Öner et al. [20] introduced a decision support model to jointly optimize the reliability level and spare parts inventory level of a single-component system in the design phase. To maximize the profit, Huang et al. [8] proposed a model to compute the optimal warranty policy under different market situations.

In most previous work on system reliability optimization problem, all parameters are assumed to be precise. However in reality there is considerable uncertainty and inaccuracy in the estimation of the model parameters, especially the component reliabilities. To measure the uncertain system reliability due to the insufficient component-level failure data, Coit [5], Jin and Coit [12] and Ramirez-Marquez and Levitin [22] estimated the confidence intervals of the system reliability for different systems. To solve the reliability optimization problem

considering component reliability uncertainty, Hussain and Murthy [9] developed a model to determine the optimal redundancy design considering quality uncertainty due to manufacture variability, and obtained the best tradeoff between the manufacturing cost and warranty cost. In Hussain and Murthy [10], the component uncertain reliability has been modeled by a reliability growth model, and an optimization model has been proposed to find the best tradeoff between the development cost and the warranty cost. Feizollahi and Modarres [7] proposed a reliability design framework to address the uncertain component reliability by using the Min-Max regret (also known as robust deviation) approach and transformed the nonlinear programming formulation to a linear binary version to get the exact solutions. None of the above works considered interval availability under a PBC.

In this paper, we focus on the cases in which all three aspects in Figure 1 are considered. We propose a reliability optimization model with uncertain component reliabilities and approximately evaluated the LCC under a PBC considering interval availability.

3. Model

During the design phase of a system, engineers have to select a certain design from all the possible alternatives for each critical component in the system. Suppose the system is comprised of a set of critical components $I = \{1, 2, \ldots, |I|\}$. If one of these critical components fails, the system as a whole stops working. For each critical component , one design needs to be selected from possible alternatives denoted by $J_i = \{1, 2, ..., |J_i|\}$. Each design candidate in J_i has its own uncertain reliability parameters and cost parameters. We aim to find out the optimal combination of designs for the system to minimize the expected LCC over the service period T of a PBC contract.

We assume that the failure process of each of design from each component is independent and follows a Poisson process. Then the lifetimes of the components are also independent and exponentially distributed. Then for a certain rough design j of component i ($i \in I$, $j \in J_i$), we denote its failure rate as Λ_{ij}. The outcome of any development process for a certain design is uncertain. Therefore, the failure rate Λ_{ij} of design j for component i is usually not known for sure before the development of the rough design. We use $f_{\Lambda_{ij}}(.)$ as the probability density function of the random failure rate Λ_{ij} before the development of the rough design, which reflects the prior belief/information about the reliability uncertainty of the technologies used in the rough design. In the evaluation of the expected LCC over the service period T, these design uncertainties will be taken into account for different combinations of rough designs.

The system will be sold together with a PBC contract over a service period $[0, T]$. The

OEM is responsible for all the repairs in $[0, T]$. Moreover the total downtime of the service period should be lower than a predetermined targeted downtime D_0. A penalty cost will be paid by the OEM to compensate the customer if the total downtime exceeds D_0. As a result, the expected total cost of a system over $[0, T]$ consists of three parts: (a) acquisition cost, (b) repair cost, and (c) penalty cost. A detailed description of the evaluation of these cost elements are given in the following subsections.

Define the binary decision variable x_{ij} as

$$x_{ij} = \begin{cases} 1 & \text{if rough design } j \text{ for component } i \text{ is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

We assume the OEM can only select one design from all the possible candidates for each component, so we have $\sum_{j=1}^{|J_i|} x_{ij} = 1$. The vector of x_{ij} of a given component i, $\boldsymbol{x}_i = [x_{i1}, x_{i2}, \ldots, x_{i|J_i|}]$ represents which component design has been chosen for component i. While the vector of $\boldsymbol{x}_i$ $(i \in I)$, $\boldsymbol{x} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{|I|}]$ is the decision variable of the system representing the selection of the alternative design for each component. The OEM is interested in minimizing the expected LCC, $\pi(\boldsymbol{x})$, which is the sum of the total design cost $A(\boldsymbol{x})$, the expected system repair cost $R(\boldsymbol{x})$ and the expected penalty cost $P(\boldsymbol{x})$. Due to the randomness of the failure rates $\Lambda_1(\boldsymbol{x}_1), \Lambda_2(\boldsymbol{x}_2), ..., \Lambda_{|I|}(\boldsymbol{x}_{|I|})$ in rough designs, the total life cycle cost is random, so we only consider the expected value of the total life cycle cost $\pi(\boldsymbol{x})$. If the decision maker is risk-neutral, the optimization model of this problem can be formulated as:

$$\begin{aligned} \text{(P)} \qquad & min_{\boldsymbol{x}} \qquad \pi(\boldsymbol{x}) \\ & \text{s.t.} \qquad \sum_{j=1}^{|J_i|} x_{ij} = 1, \qquad \text{for all } i \in I \\ & \qquad\quad x_{ij} \in \{0, 1\}, \qquad \text{for all } i \in I, j \in J_i \end{aligned}$$

where $\pi(\boldsymbol{x}) = A(\boldsymbol{x}) + R(\boldsymbol{x}) + P(\boldsymbol{x})$. This optimization problem is difficult to solve, because of the complicated form of the objective function. In the objective function, the expected penalty cost $P(\boldsymbol{x})$ is a multiple integration over the ranges of the random failure rates, which is difficult to calculate. Therefore, in the next section, an approximation method will be proposed to make the evaluation of the objective function easier.

4. Evaluation

In this section we will introduce two ways to evaluate the related costs. We will first present the exact evaluation of $A(\boldsymbol{x})$, $R(\boldsymbol{x})$ and $P(\boldsymbol{x})$, given that the exact evaluation of $P(\boldsymbol{x})$ is computationally intractable when $|I|$ is large. We also propose three different methods to evaluate $P(\boldsymbol{x})$ approximately, the zero-uncertainty method, the partial-uncertainty method, and the full-uncertainty method. Each differs in considering the level of uncertainty of $P(\boldsymbol{x})$.

4.1. Exact evaluation

In this subsection, we describe how to evaluate a given policy exactly. For the exact evaluation of a given policy $\boldsymbol{x}$, we present the exact evaluation of $A(\boldsymbol{x})$, $R(\boldsymbol{x})$, $P(\boldsymbol{x})$ accordingly.

4.1.1. Acquisition cost

Let c_{ij}^a denote the cost of designing and manufacturing component i according to a rough design j ($i \in I$, $j \in J_i$). It includes all costs incurred to realize a certain rough design of component during the design phase, e.g., human resources, experimental equipment, testing or prototype units, etc. The acquisition cost for component i is given by

$$A_i(\boldsymbol{x}_i) = \sum_{j=1}^{|J_i|} c_{ij}^a x_{ij},$$

where $\boldsymbol{x}_i = [x_{i1}, x_{i2}, ..., x_{i|J_i|}]$ represents the selection of rough designs for component i, and the total system acquisition cost is given by

$$A(\boldsymbol{x}) = \sum_{i=1}^{|I|} A_i(\boldsymbol{x}_i) = \sum_{i=1}^{|I|} \sum_{j=1}^{|J_i|} c_{ij}^a x_{ij}, \tag{1}$$

where $\boldsymbol{x}$ represents the selection plan of rough designs for all the critical components in the system.

4.1.2. Repair cost

When a failure occurs in period $[0, T]$, a repair will be performed by the OEM. We assign c_{ij}^r as the repair cost for each failure of the jth rough design for component i ($i \in \{1, 2, ..., |I|\}$, $j \in \{1, 2, ..., |J_i|\}$). The repair cost c_{ij}^r corresponds to diagnosis cost, replacement cost, and other service costs for each repair. A failure-based policy for maintenance is assumed for this multi-component system in order to evaluate the maintenance cost over the service period T. The evaluation of repair cost based on a failure-based policy is relatively accurate and conservative. Let $S_i(\boldsymbol{x}_i)$ denote the total number of failures for component i during $[0, T]$.

Under such a failure-based policy and the assumption that the lifetimes of components are exponentially distributed, the expected number of repairs for component i during $[0, T]$, $\mathbb{E}[S_i(\boldsymbol{x}_i)]$ is given by:

$$\mathbb{E}\Big[S_i(\boldsymbol{x}_i)\Big] = \mathbb{E}\Big[\Lambda_i(\boldsymbol{x}_i)T\Big] = \sum_{j=1}^{|J_i|} \mathbb{E}(\Lambda_{ij}T)x_{ij} = \sum_{j=1}^{|J_i|} \mu_{ij}Tx_{ij},$$

where Λ_{ij} is the failure rate for the jth design of component i, while μ_{ij} and σ_{ij} are the expectation and standard deviation of Λ_{ij}. $\Lambda_i(\boldsymbol{x}_i) = \sum_{j=1}^{|J_i|} \Lambda_{ij}x_{ij}$ is the random failure rate of component i given a certain rough design $\boldsymbol{x}_i$. Then the expected repair cost for component i, $R_i(\boldsymbol{x}_i)$ is expressed by:

$$R_i(\boldsymbol{x}_i) = \sum_{j=1}^{|J_i|} \mathbb{E}(\Lambda_{ij}Tc_{ij}^r)x_{ij} = \sum_{j=1}^{|J_i|} \mu_{ij}Tc_{ij}^r x_{ij}.$$

Given that the failure processes of all the critical components are independent of each other, the expected system repair cost $R(\boldsymbol{x})$ in $[0, T]$ is given as:

$$R(\boldsymbol{x}) = \sum_{i=1}^{n} R_i(\boldsymbol{x}_i) = \sum_{i=1}^{|I|} \sum_{j=1}^{|J_i|} \mu_{ij}Tc_{ij}^r x_{ij}. \tag{2}$$

4.1.3. Penalty cost

A fixed period of system downtime $r_{ij}(r_{ij} << T)$ will be incurred due to a random failure of component i with rough design j in the system ($i \in \{1, 2, ..., |I|\}$, $j \in \{1, 2, ..., |J_i|\}$). Notice that while evaluating the repair costs in the previous section we ignore the downtime since the downtime is usually negligible compared with the service period T, for the same reason, when we computing the penalty cost, we ignore the downtime as well. However, under a PBC contract, when the total system downtime over the service period T exceeds D_0, a penalty cost should be paid by the OEM to customers with a rate c^p. D_0 is at the same time scale as r_{ij}. We assume the system downtime of each failure varies among different components with different rough designs. Hence the total system downtime $D(\boldsymbol{x})$ over the service period T depends on the number of failures $S_i(\boldsymbol{x}_i)$ and the repair time per failure $r_i(\boldsymbol{x}_i)$ for component $i \in \{1, \ldots, |I|\}$ in $[0, T]$. The repair time per failure for component i, $r_i(\boldsymbol{x}_i) = \sum_{j=1}^{|J_i|} r_{ij}x_{ij}$, is a fixed value after the selection plan for component i has been made. Notice that the number of failures from component i, $S_i(\boldsymbol{x}_i)$, is a Poisson distributed random variable, and we neglect r_{ij} in calculating $S_i(\boldsymbol{x}_i)$ in $[0, T]$, then the distribution of

$S_i(\boldsymbol{x}_i)$ is given as

$$\begin{aligned} Pr\Big[S_i(\boldsymbol{x}_i) = s_i\Big] &= \sum_{j=0}^{|J_i|} x_{ij} Pr(S_{ij} = s_i) \\ &= \sum_{j=0}^{|J_i|} x_{ij} \int_0^{\infty} Pr(S_{ij} = s_i | \Lambda_{ij} = \lambda_{ij}) f_{\Lambda_{ij}}(\lambda_{ij}) d\lambda_{ij} \\ &= \sum_{j=0}^{|J_i|} x_{ij} \int_0^{\infty} \frac{e^{-\lambda_{ij}T}(\lambda_{ij}T)^{s_i}}{s_i!} f_{\Lambda_{ij}}(\lambda_{ij}) d\lambda_{ij}. \end{aligned}$$

According to the service contract, the OEM should pay a penalty cost to the customer. Since the failure rates $\Lambda_1(\boldsymbol{x}_1), \Lambda_2(\boldsymbol{x}_2), ..., \Lambda_n(\boldsymbol{x}_n)$ are random variables, the expected penalty cost is a random variable as well. Without loss of generality, we assume the failure rates are continuous random variables, with probability density functions $f_{\Lambda_{ij}}(.)$ over region $\mathcal{O}_{ij}$. Then the expected penalty cost due to extra downtime exceeding D_0 is given as

$$\begin{aligned} P(\boldsymbol{x}) &= \mathbb{E}\Big\{\Big[D(\boldsymbol{x}) - D_0\Big]^+ c^p\Big\} \\ &= \sum_{s_1=0}^{\infty}\sum_{s_i=0}^{\infty}\cdots\sum_{s_{|I|}=0}^{\infty}\prod_{i=1}^{|I|} Pr\Big[S_i(\boldsymbol{x}_i) = s_i\Big]\Big[\sum_{i=1}^{|I|} s_i r_i(\boldsymbol{x}_i) - D_0\Big]^+ c^p \\ &= \sum_{s_1=0}^{\infty}\sum_{s_i=0}^{\infty}\cdots\sum_{s_{|I|}=0}^{\infty}\prod_{i=1}^{|I|}\sum_{j=1}^{|J_{|i|}|} x_{ij}\int_0^{\infty}\frac{e^{-\lambda_{ij}T}(\lambda_{ij}T)^{s_i}}{s_i!} f_{\Lambda_{ij}}(\lambda_{ij}) d\lambda_{ij}\Big[\sum_{i=1}^{|I|} s_i r_i(\boldsymbol{x}_i) - D_0\Big]^+ c^p \quad (3) \end{aligned}$$

4.2. Approximate evaluation

In this section, we are going to explain how to approximately evaluate a given selection plan of rough designs for all the component in the system. For a given policy $\underline{\boldsymbol{x}} = [\underline{\boldsymbol{x}}_1, \underline{\boldsymbol{x}}_2, \ldots, \underline{\boldsymbol{x}}_{|I|}]$, the design cost $A(\underline{\boldsymbol{x}})$ and the expected repair cost $R(\underline{\boldsymbol{x}})$ can be determined from (1) and (2). These two cost terms are linear functions of the decision variables $\underline{\boldsymbol{x}} = [\underline{\boldsymbol{x}}_1, \underline{\boldsymbol{x}}_2, \ldots, \underline{\boldsymbol{x}}_{|I|}]$.

For the exact evaluation of the expected penalty cost $P(\underline{\boldsymbol{x}})$, we will suffer from the "curse of dimensionality" when the number of critical components becomes large, since each critical component contributes a dimension in computing the convolution and integration in (3). The computation time will explode as the number of critical components grows. From (3), we can also observe that there are two levels of uncertainty existing in $D(\underline{\boldsymbol{x}})$ throughout $[0, T]$. The first level of uncertainty originates from the number of failures $S_i(\underline{\boldsymbol{x}})$ ($i \in \{1, 2, ..., |I|\}$), which is a Poisson distributed random variable. The second level of

uncertainty comes from the failure rates λ_{ij} ($i \in \{1,2,\ldots,|I|\}$, $j \in \{1,2,\ldots,|J_i|\}$), which is a generally distributed random variable. We now introduce three approximation methods to estimate the expected penalty cost, which vary in dealing with the uncertainty levels of $D(\underline{\boldsymbol{x}})$. In the first commonly used evaluation method, we use the expected the downtime as the actual downtime, thus ignoring both levels of the uncertainty. In the current literature, for example, the uncertainty existing in the failure rates are usually ignored, so in the second evaluation method, we only consider the uncertainty of the number of failures of the system. While for the third evaluation method, we address both uncertainty levels of $D(\underline{\boldsymbol{x}})$ by using a full-uncertainty method. These methods will be compared in Section 5.

To derive the first and second moment of $D(\underline{\boldsymbol{x}})$, for the total downtime of the jth design in component i in $[0,T]$, the first moment is given as:

$$\begin{aligned}
\mathbb{E}(D_{ij}) &= \int_{\lambda_{ij}\in\mathcal{O}_{ij}} \mathbb{E}\Big[D_{ij}\Big|\Lambda_{ij}=\lambda_{ij}\Big] f_{\Lambda_{ij}}(\lambda_{ij})d\lambda_{ij} \\
&= \int_{\lambda_{ij}\in\mathcal{O}_{ij}} \lambda_{ij} T r_{ij} f_{\Lambda_{ij}}(\lambda_{ij})d\lambda_{ij} \\
&= \mu_{ij} T r_{ij}.
\end{aligned}$$

By using the moment generating function of D_{ij}, we obtain:

$$\begin{aligned}
\mathbb{E}(D_{ij}^2) &= \int_{\lambda_{ij}\in\mathcal{O}_{ij}} \mathbb{E}\Big[D_{ij}^2\Big|\Lambda_{ij}=\lambda_{ij}\Big] f_{\Lambda_{ij}}(\lambda_{ij})d\lambda_{ij}, \\
&= \int_{\lambda_{ij}\in\mathcal{O}_{ij}} r_{ij}^2 \lambda_{ij} T(1+\lambda_{ij}T) f_{\Lambda_{ij}}(\lambda_{ij})d\lambda_{ij}, \\
&= r_{ij}^2 T\mu_{ij} + r_{ij}^2 T^2(\sigma_{ij}^2+\mu_{ij}^2), \\
\mathrm{Var}(D_{ij}) &= \mathbb{E}\Big[D_{ij}^2\Big] - \Big(\mathbb{E}[D_{ij}]\Big)^2 = r_{ij}^2[\mu_{ij}T + T^2\sigma_{ij}^2]. \qquad (4)
\end{aligned}$$

Then, for a single component i, we have:

$$\begin{aligned}
D_i(\underline{\boldsymbol{x}}_i) &= \sum_{j=1}^{|J_i|} D_{ij}\underline{x}_{ij}, \\
\mathbb{E}[D_i(\underline{\boldsymbol{x}}_i)] &= \sum_{j=1}^{|J_i|} r_{ij}\mu_{ij}T\underline{x}_{ij}, \\
\mathrm{Var}[D_i(\underline{\boldsymbol{x}}_i)] &= \sum_{j=1}^{|J_i|} r_{ij}^2[T^2\sigma_{ij}^2 + T\mu_{ij}]\underline{x}_{ij}. \qquad (5)
\end{aligned}$$

Finally, for the total downtime of the system in $[0, T]$:

$$\begin{aligned} D(\underline{\boldsymbol{x}}) &= \sum_{i=1}^{|I|} D_i(\underline{\boldsymbol{x}}_i) = \sum_{i=1}^{|I|}\sum_{j=1}^{|J_i|} D_{ij}\underline{x}_{ij}, \\ \mu_D(\underline{\boldsymbol{x}}) &= \mathbb{E}[D(\underline{\boldsymbol{x}})] = \sum_{i=1}^{|I|} \mathbb{E}[D_i(\underline{\boldsymbol{x}}_i)] = \sum_{i=1}^{|I|}\sum_{j=1}^{|J_i|} r_{ij}\mu_{ij}T\underline{x}_{ij}, \qquad (6) \\ \sigma_D^2(\underline{\boldsymbol{x}}) &= \mathrm{Var}[D(\underline{\boldsymbol{x}})] = \sum_{i=1}^{|I|}\sum_{j=1}^{|J_i|} \mathrm{Var}[D_i(\underline{\boldsymbol{x}}_i)] = \sum_{i=1}^{|I|}\sum_{j=1}^{|J_i|} r_{ij}^2[T^2\sigma_{ij}^2 + T\mu_{ij}]\underline{x}_{ij}. \qquad (7) \end{aligned}$$

4.2.1. Zero-uncertainty method

The most direct way of approximating the system downtime is to use expected downtime instead of the actual downtime. According to (6), zero-uncertainty method use $\mathbb{E}[D(\underline{\boldsymbol{x}})]$ as $D(\underline{\boldsymbol{x}})$, then we have the first evaluation of system downtime $D_{A_1}(\underline{\boldsymbol{x}})$ described as:

$$D_{A_1}(\underline{\boldsymbol{x}}) = \sum_{i=1}^{|I|}\sum_{j=1}^{|J_i|} r_{ij}\mu_{ij}T\underline{x}_{ij},$$

and the expected excess downtime $D^E(\underline{\boldsymbol{x}})$ and penalty cost $P(\underline{\boldsymbol{x}})$ can be approximated as:

$$\begin{aligned} D_{A_1}^E(\underline{\boldsymbol{x}}) &= \Big[D_{A_1}(\underline{\boldsymbol{x}}) - D_0\Big]^+. \\ P_{A_1}(\underline{\boldsymbol{x}}) &= \Big[D_{A_1}(\underline{\boldsymbol{x}}) - D_0\Big]^+ c^p \end{aligned}$$

4.2.2. Partial-uncertainty method

In the current literature, for example in Mettas and Kallenberg [16], Öner et al. [20] and Huang et al. [8], they do not consider the uncertainty of λ_{ij} ($i \in I$, $j \in |J_i|$). So for the second approximation evaluation method, given that the lifetimes of the components are exponentially distributed, we will simply assume λ_{ij} ($i \in I$, $j \in |J_i|$) is a constant which equals to its mean. Given that $D(\underline{\boldsymbol{x}})$ is the summation of $D_i(\underline{\boldsymbol{x}}_i)$, and the number of failures for each component is poisson distributed, we approximate $D^E(\underline{\boldsymbol{x}})$ by fitting a mixed Erlang distribution to the first and second moments of $D(\underline{\boldsymbol{x}})$. The first moment of $D(\underline{\boldsymbol{x}})$, $\mu_D(\underline{\boldsymbol{x}})$, is given in (6). Under the assumption that $\lambda_{ij} = \mu_{ij}$ and $\sigma_{ij} = 0$, $\mathrm{Var}(D_{ij})$, $\mathrm{Var}[D_i(\underline{\boldsymbol{x}}_i)]$ and

$\sigma_D^2(\underline{\boldsymbol{x}})$ in (4), (5) and (7) are rewritten as:

$$\begin{aligned}
\mathrm{Var}(D'_{ij}) &= r_{ij}^2 \mu_{ij} T, \\
\mathrm{Var}[D'_i(\underline{\boldsymbol{x}}_i)] &= \sum_{j=1}^{|J_i|} r_{ij}^2 T \mu_{ij} \underline{x}_{ij}, \\
\sigma'^2_D(\underline{\boldsymbol{x}}) &= \sum_{i=1}^{|I|} \sum_{j=1}^{|J_i|} r_{ij}^2 T \mu_{ij} \underline{x}_{ij}.
\end{aligned} \tag{8}$$

From (6) and (8), the coefficient of variation of $D(\underline{\boldsymbol{x}})$ is given as:

$$c'_v(\underline{\boldsymbol{x}}) = \frac{\sigma'_D(\underline{\boldsymbol{x}})}{\mu_D(\underline{\boldsymbol{x}})} = \frac{\sqrt{\sum_{i=1}^{|I|} \sum_{j=1}^{|J_i|} r_{ij}^2 \mu_{ij} T \underline{x}_{ij}}}{\sum_{i=1}^{|I|} \sum_{j=1}^{|J_i|} r_{ij} \mu_{ij} T \underline{x}_{ij}} \tag{9}$$

Given that $\mu_D(\underline{\boldsymbol{x}}) > 1$, if $0 < c'_v(\underline{\boldsymbol{x}}) \leq 1$, according to Tijms [24](335-337), we fit the downtime distribution to an Erlang$(k'-1, k')$ distribution with parameters(k', θ', q'_E) such that the first two moments of $D(\boldsymbol{x})$ match the first two moments of the Erlang $(k'-1, k')$ distribution. Thus the parameters of the Erlang $(k'-1, k')$ distribution can be obtained as:

$$k'(\underline{\boldsymbol{x}}) = \lceil \frac{1}{c'^2_v(\underline{\boldsymbol{x}})} \rceil, \tag{10}$$

$$q'_E(\boldsymbol{x}) = \frac{1}{1 + c'^2_v(\boldsymbol{x})} \left[k'(\boldsymbol{x}) c'^2_v(\boldsymbol{x}) - \sqrt{k'(\boldsymbol{x}) \big[1 + c'^2_v(\boldsymbol{x})\big] - k'^2(\boldsymbol{x}) c'^2_v(\boldsymbol{x})} \right], \tag{11}$$

$$\theta'(\boldsymbol{x}) = \frac{k'(\boldsymbol{x}) - q'_E(\boldsymbol{x})}{\mu'_D(\boldsymbol{x})}. \tag{12}$$

Then the expected excess downtime can be approximated as:

$$D^E_{A_2}(\boldsymbol{x}) = \frac{q'_H(\boldsymbol{x})}{\theta'_1(\boldsymbol{x})} e^{-\theta'_1(\boldsymbol{x}) D_0} + \frac{1 - q'_H(\boldsymbol{x})}{\theta'_2(\boldsymbol{x})} e^{-\theta'_2(\boldsymbol{x}) D_0}. \tag{13}$$

The derivation can be found in Appendix B.2.

If $c'_v \geq 1$, we fit the downtime distribution to an Hyperexponential distribution with parameters $(\theta'_1, \theta'_2, q'_H)$ such that the first two moments of $D(\boldsymbol{x})$ match the first two moments of the Hyperexponential distribution. Thus the parameters of the Hyperexponential

distribution can be obtained as:

$$\theta_1'(\boldsymbol{x}) = \frac{2}{\mu_D(\boldsymbol{x})}\left(1+\sqrt{\frac{c_v'^2(\boldsymbol{x})-\frac{1}{2}}{c_v'^2(\boldsymbol{x})+1}}\right), \tag{14}$$

$$\theta_2'(\boldsymbol{x}) = \frac{4}{\mu_D(\boldsymbol{x})} - \theta_1'(\boldsymbol{x}), \tag{15}$$

$$q_H'(\boldsymbol{x}) = \frac{\theta_1'(\boldsymbol{x})(\theta_2'(\boldsymbol{x})\mu_D(\boldsymbol{x})-1)}{\theta_2'(\boldsymbol{x})-\theta_1'(\boldsymbol{x})}. \tag{16}$$

Then $D^E(\underline{\boldsymbol{x}})$ can be approximated by:

$$D_{A_2}^E(\underline{\boldsymbol{x}}) = \frac{q_H'(\boldsymbol{x})}{\theta_1'(\boldsymbol{x})}e^{-\theta_1'(\boldsymbol{x})D_0} + \frac{1-q_H'(\boldsymbol{x})}{\theta_2'(\boldsymbol{x})}e^{-\theta_2'(\boldsymbol{x})D_0}, \tag{17}$$

$$\tag{18}$$

The derivation can be found in Appendix B.2. And $P(\underline{\boldsymbol{x}})$ can be approximated by:

$$P_{A_2}(\underline{\boldsymbol{x}}) = D_{A_2}^E(\underline{\boldsymbol{x}})c^p. \tag{19}$$

4.2.3. Full-uncertainty method

To approximate both the uncertainty of the failure rate as well as the number of failures, then the number of failures for each component is Poisson distributed with an uncertain parameter. We approximate the downtime by two-moment fit method as well. We get the first moment and variance of $D(\underline{\boldsymbol{x}})$, from (6) and (7). Then the coefficient of variation of $D(\underline{\boldsymbol{x}})$ is given by:

$$c_v(\boldsymbol{x}) = \frac{\sigma_D(\boldsymbol{x})}{\mu_D(\boldsymbol{x})} = \frac{\sqrt{\sum_{i=1}^{|I|}\sum_{j=1}^{|J_i|} r_{ij}^2[T^2\sigma_{ij}^2 + T\mu_{ij}]\underline{x}_{ij}}}{\sum_{i=1}^{|I|}\sum_{j=1}^{|J_i|} r_{ij}\mu_{ij}T\underline{x}_{ij}}.$$

Under the condition that $\mu_D > 0$, and $0 < c_v \leq 1$, we fit the downtime distribution to an Erlang$(k-1,k)$ distribution with parameters(k,θ,q_E). Thus, similar to the partial-uncertainty method the parameters of the Erlang $(k-1,k)$ distribution can be obtained as

:

$$k(\boldsymbol{x}) = \lceil \frac{1}{c_v^2(\boldsymbol{x})} \rceil, \tag{20}$$

$$q_E(\boldsymbol{x}) = \frac{1}{1+c_v^2(\boldsymbol{x})}\left[k(\boldsymbol{x})c_v^2(\boldsymbol{x}) - \sqrt{k(\boldsymbol{x})\big[1+c_v^2(\boldsymbol{x})\big] - k^2(\boldsymbol{x})c_v^2(\boldsymbol{x})}\right], \tag{21}$$

$$\theta(\boldsymbol{x}) = \frac{k(\boldsymbol{x}) - q_E(\boldsymbol{x})}{\mu_D(\boldsymbol{x})}. \tag{22}$$

Then for random failure rates $\Lambda_1(\boldsymbol{x_1})$, $\Lambda_2(\boldsymbol{x_2})$,..., $\Lambda_n(\boldsymbol{x_n})$, $D^E(\underline{\boldsymbol{x}})$ can be approximated by

$$\begin{aligned} D_{A_3}^E(\boldsymbol{x}) &= \left[\frac{k(\boldsymbol{x}) - q_E(\boldsymbol{x})}{\theta(\boldsymbol{x})} - D_0\right] \sum_{j=0}^{k(\boldsymbol{x})-2} \frac{\big[\theta(\boldsymbol{x})D_0\big]^j}{j!} e^{-\theta(\boldsymbol{x})D_0} \\ &\quad + \left[\frac{k(\boldsymbol{x}) - q_E(\boldsymbol{x})}{\theta(\boldsymbol{x})}\right] \frac{\big[\theta(\boldsymbol{x})D_0\big]^{k(\boldsymbol{x})-1}}{\big[k(\boldsymbol{x})-1\big]!} e^{-\theta(\boldsymbol{x})D_0}. \end{aligned} \tag{23}$$

The derivation is presented in Appendix B.1.

If $c_v \geq 1$, we fit the downtime distribution to an Hyperexponential distribution with parameters $(\theta_1, \theta_2, q_H)$ such that the first two moments of $D(\boldsymbol{x})$ match the first two moments of the Hyperexponential distribution. Thus the parameters of the Hyperexponential distribution can be obtained as:

$$\theta_1(\boldsymbol{x}) = \frac{2}{\mu_D(\boldsymbol{x})}\left(1 + \sqrt{\frac{c_v^2(\boldsymbol{x}) - \frac{1}{2}}{c_v^2(\boldsymbol{x})+1}}\right), \tag{24}$$

$$\theta_2(\boldsymbol{x}) = \frac{4}{\mu_D(\boldsymbol{x})} - \theta_1(\boldsymbol{x}), \tag{25}$$

$$q_H(\boldsymbol{x}) = \frac{\theta_1(\boldsymbol{x})(\theta_2(\boldsymbol{x})\mu_D(\boldsymbol{x}) - 1)}{\theta_2(\boldsymbol{x}) - \theta_1(\boldsymbol{x})}. \tag{26}$$

Then $D^E(\underline{\boldsymbol{x}})$ can be approximated by

$$D_{A_3}^E(\boldsymbol{x}) = \frac{q_H(\boldsymbol{x})}{\theta_1(\boldsymbol{x})} e^{-\theta_1(\boldsymbol{x})D_0} + \frac{1 - q_H(\boldsymbol{x})}{\theta_2(\boldsymbol{x})} e^{-\theta_2(\boldsymbol{x})D_0}. \tag{27}$$

The derivation can be found in Appendix B.2.And $P(\underline{\boldsymbol{x}})$ of the full-uncertainty method can approximated by:

$$P_{A_3}^E(\boldsymbol{x}) = D_{A_3}^E(\boldsymbol{x}) c^p \tag{28}$$

The procedure of the two-moment fit method is summarized in the following algorithm.

Algorithm 1

Step 1 Compute the $\mu_D(\boldsymbol{x})$, $\sigma_D^2(\boldsymbol{x})$ and $c_v(\boldsymbol{x})$ of the downtime distribution, check the value of $c_v(\boldsymbol{x})$: if $0 < c_v(\boldsymbol{x}) \leq 1$, go to step 2; if $c_v(\boldsymbol{x}) > 1$, go to step 3.

Step 2 Fit the first two moments of an Erlang$(k-1, k)$ distribution to be equal to the first two moments of the downtime distribution.

Step 2-a Let $k(\boldsymbol{x})$, $\theta(\boldsymbol{x})$ and $q_E(\boldsymbol{x})$ be the parameters of the fitted Erlang$(k-1, k)$ distribution, and compute the values of $k(\boldsymbol{x})$, $\theta(\boldsymbol{x})$ and $q_E(\boldsymbol{x})$ according to (10)-(12) for the partial-uncertainty method, or (20)-(22) full-uncertainty method.

Step 2-b Calculate $D_{A_2}^E(\boldsymbol{x})$ according to (13) and $D_{A_3}^E(\boldsymbol{x})$ according to (23).

Step 3 Fit the first two moments of a Hyperexponential distribution to be equal to the first two moments of the downtime distribution.

Step 3-a Let $\theta_1(\boldsymbol{x})$, $\theta_2(\boldsymbol{x})$ and $q_H(\boldsymbol{x})$ be the parameters of the fitted Hyperexponential distribution, compute the values of these parameters according to (14)-(16) for the partial-uncertainty method and (24)-(26) for the full-uncertainty method.

Step 3-b Compute $D_{A_2}^E(\boldsymbol{x})$, $D_{A_3}^E(\boldsymbol{x})$ according to (17) and (27).

After approximate evaluation, our optimization problem is reformulated as P′:

$$
\begin{array}{lllll}
(\mathrm{P}') & min_{\boldsymbol{x}} & \pi_{A_k}(\boldsymbol{x}) & & k = 1, 2, 3. \\
& \text{s.t.} & \displaystyle\sum_{j=1}^{|J_i|} x_{ij} = 1, & & \text{for all } i \in I \\
& & x_{ij} \in \{0, 1\}, & & \text{for all } i \in I, j \in J_i
\end{array}
$$

where $\pi(\boldsymbol{x}) = A(\boldsymbol{x}) + R(\boldsymbol{x}) + P_{A_k}(\boldsymbol{x})$.

5. Computational results

In this section, we propose two numerical experiments to test the accuracy of our evaluation methods by comparing the results of these methods with the simulation results from different test beds. In section 5.1, we investigate the quality of the evaluation results generated by the zero-uncertainty method, partial-uncertainty method, and full-uncertainty

method, and compare them with the results generated by a simulation procedure mentioned in Appendix A. In section 5.2, we investigate the quality of the optimal policies generated by each evaluation method together with full enumeration.

5.1. Accuracy of the approximation

In this section, we execute a numerical experiment to assess the accuracy of the three evaluation methods mentioned in the Section 4.2. To investigate the effect of three factors, we conduct a full factorial test bed of 210 instances.

5.1.1. Test bed

To assess the quality of these methods, we compare the evaluation results of these methods with the simulation results. Our simulation Monte Carlo simulation method is described in Appendices A. D_S^E is computed, as well as the gaps between D_S^E and $D_{A_1}^E$, $D_{A_2}^E$ and $D_{A_3}^E$. A full factorial test bed is set up to show the accuracy of the evaluation procedures under different parameter settings. We identify three main factors: the number of the critical components n, coefficient of variation c_v of $\lambda_i(\underline{\boldsymbol{x}}_i)$ and the downtime budget factor D_f. Notice that $\lambda_i(\underline{\boldsymbol{x}}_i)$, $\mu_i(\underline{\boldsymbol{x}}_i)$ and $r_i(\underline{\boldsymbol{x}}_i)$ are functions of the given design $\underline{\boldsymbol{x}}_i$. We will treat them as parameters λ_i, μ_i and r_i in the evaluation. Similarly, $\sigma_i(\underline{\boldsymbol{x}}_i)$ is also considered as a parameter σ_i computed by $c_v\mu_i$. We generated 210 instances by taking all combinations of the factor values shown in Table 1.

Table 1: The parameter setting of the test bed

n	c_v	D_f
5, 25, 50, 75, 100	0.2, 0.5, 0.8, 1.1, 1.4, 1.7	1, 1.05, 1.1, 1.15, 1.2, 1.25, 1.3

In the test bed, the factor D_f is a coefficient to generate different values of D_0 by the following expression,

$$D_0 = D_f \sum_{i=1}^{n} \mu_i r_i T, \tag{29}$$

where $\sum_{i=1}^{n} \mu_i r_i T$ is the expected total downtime. Thus, if D_f is one, the targeted downtime D_0 is set to be equal to the expected total downtime. The setting of the fixed parameters in the test bed is given in Table 2. The repair time r_i varies for different i by taking a value from $\{1, 3, 5\}$. The average failure rate μ_i varies for different i as a sequence. We assume λ_i $(i \in |I|)$ is lognormally distributed.

For the zero-uncertainty method, it is apparent that when $D_f \geq 1$, $D^E_{A_1} = 0$ for all the 210 instances. For the fixed-failure rate method, since the randomness of the failure rates are not considered, $D^E_{A_2}$ is only related to n and D_f, and only 35 instances are computed, to compare the gap between $D^E_{A_2}$ and D^E_S under all the 210 parameter settings. Namely, we can get one value of $D^E_{A_2}$ for different c_v under the same settings of (n, D_f).

Table 2: Parameter values for μ_i and r_i, $i = \{1, 2, ..., n\}$

n	T	μ_i	r_i
5, 25, 50, 75, 100	10	$\{0.200, ..., 0.200 - (i-1)\frac{0.18}{n-1}, ..., 0.02\}$	$\{1, 3, 5, 1, 3, 5, ...\}$

5.1.2. Results and managerial insights

We summarize the results of the test bed in Table 3. To see the deviations of the approximation results from the simulation results for each instance, we first compute the proportion of D^E in D_0, e.g., $D^E_{A_1}/\sum_{i=1}^n \mu_i r_i T$ for the approximation methods and $D^E_S/\sum_{i=1}^n \mu_i r_i T$ for the simulation methods, as well as the confidence intervals of the simulation results. The confidence intervals of all the simulation results are relatively small, which shows the accuracy of our simulation results. Then the accuracy of our approximation is assessed by the gaps between $\{D^E_{A_1}/\sum_{i=1}^n \mu_i r_i T,\ D^E_{A_2}/\sum_{i=1}^n \mu_i r_i T,\ D^E_{A_3}/\sum_{i=1}^n \mu_i r_i T\}$ and $D^E_S/\sum_{i=1}^n \mu_i r_i T$. The *Avg Gap* and *max Gap* columns show the average gap and maximum gap of each method for a group of instances that has the same value of one of the three factors. For example, the values 1.7%, 5.5% in the first row are the average and maximum gaps of the full-uncertainty method in the 42 instances with five components in the system. For the zero-uncertainty method, the average gap is 8.2% and the max gap is 34.3% for all the instances; the average and max gap for the partial-uncertainty method is 3.5% and 16.1% respectively; for the full-uncertainty method the average gap is 0.5% and the maximum gap is 5.5%. The full-uncertainty method performs best among all the methods. The zero-uncertainty method has the worst performance, which matches our intuition. Note that the computation times of the approximation methods for all the instances are negligible compared with simulation.

The approximation methods are more accurate when n is large (e.g., 50 or 100). This is due to the fact that the total downtime is the sum of n independent random variables, and when n is large, the total downtime converges in distribution to a normal random variable, regardless of the individual underlying distributions. The accuracy of the full-uncertainty method is not sensitive to the changes of D_f, whereas the accuracy of the zero-uncertainty method increases when D_f increases or D_0 is higher. With the increase of c_v, both the

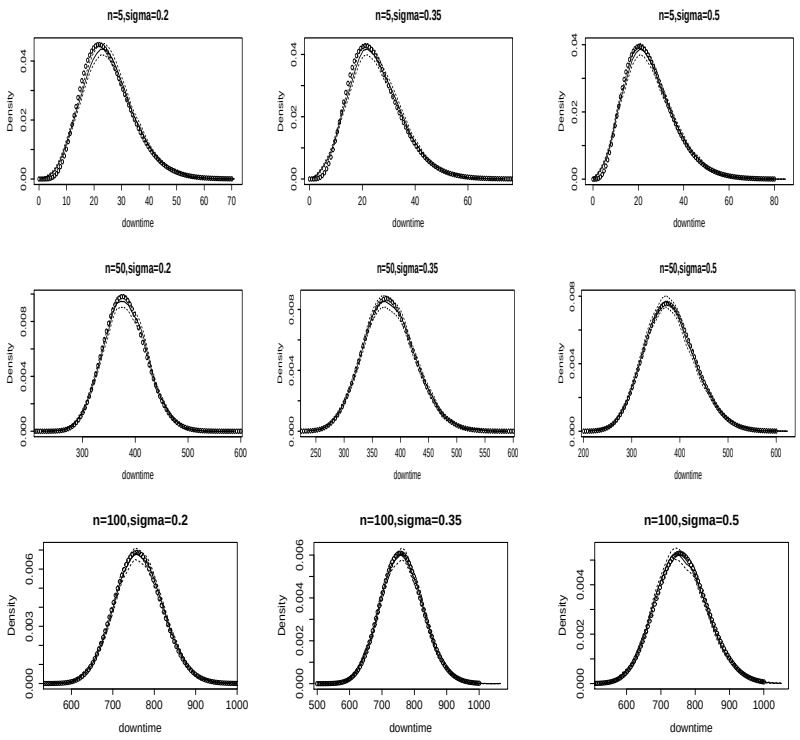

Figure 2: The simulation curve and approximation curve of the downtime density distribution under lognormal distributed failure rate

average and max gaps of each method become larger. This is because when the c_v becomes larger, σ_i becomes larger as well, which increases the uncertainty of $\lambda_i (i \in |I|)$. Given that the full-uncertainty method takes this uncertainty into account, the related gaps do not increase much, compared with the other two methods.

In order to better demonstrate the accuracy of the full-uncertainty method, we also compare the probability density functions of the downtime distributions estimated from simulation with the ones estimated from the approximation method, as shown in Figure 2. It is obvious that the downtime distributions estimated from simulation are approximately the same as the ones estimated from the full-uncertainty method for various settings of n and c_v. We also conduct another full factorial test with the same settings but under the assumption that λ_i is uniformly distributed$(i \in |I|)$. The numerical results is similar to the first test bed.

5.2. The numerical experiment on the optimization test bed

In this section, to compare the quality of the optimal policies generated by the three evaluation methods on the basis of full enumeration as well as the managerial insights of the

Table 3: Average gap and maximum gap between the simulation results and three evaluation methods

		zero-uncertainty method		partial-uncertainty method		full-uncertainty method	
		Avg Gap	*max Gap*	*Avg Gap*	*max Gap*	*Avg Gap*	*max Gap*
N	5	19.65%	34.28%	6.78%	14.53%	1.51%	5.46%
	25	7.19%	17.23%	3.36%	8.53%	0.31%	1.66%
	50	4.37%	12.64%	2.23%	6.51%	0.12%	0.80%
	75	3.20%	10.44%	1.69%	5.43%	0.08%	0.55%
	100	2.55%	9.09%	1.37%	4.76%	0.07%	0.45%
D_f	1	12.73%	34.28%	3.94%	14.53%	0.58%	5.46%
	1.05	10.15%	31.22%	3.74%	14.38%	0.51%	4.92%
	1.1	8.19%	28.48%	3.46%	14.12%	0.44%	4.47%
	1.15	6.67%	25.96%	3.12%	13.72%	0.38%	4.13%
	1.2	5.50%	23.46%	2.75%	13.02%	0.36%	4.08%
	1.25	4.60%	21.31%	2.43%	12.42%	0.34%	3.93%
	1.3	3.90%	19.70%	2.15%	12.12%	0.31%	3.48%
c_v	0.2	4.44%	20.67%	0.14%	0.91%	0.07%	0.43%
	0.5	5.22%	22.81%	0.91%	3.06%	0.06%	0.37%
	0.8	6.46%	25.73%	2.16%	5.98%	0.10%	0.45%
	1.1	7.92%	28.98%	3.61%	9.23%	0.33%	1.57%
	1.4	9.43%	32.19%	5.12%	12.44%	0.71%	3.12%
	1.7	10.87%	34.28%	6.57%	14.53%	1.23%	5.46%
All		7.39%	34.28%	3.09%	14.53%	0.42%	5.46%

optimal policies. We identify five factors and generate 243 instance from the full factorial test bed. The test bed design are described in Section 5.2.1, the comparison of optimal solutions are presented in Section 5.2.2, the managerial insights of the optimal solutions are shown in Section 5.2.3.

5.2.1. Test bed

In this subsection, we will investigate how the optimal solution changes with different parameter settings by creating a full factorial test bed. There are five factors under investigation: the downtime budget factor D_f, the acquisition cost c^a_{ij} ($i \in |I|$, $j \in |J_i|$), the expected failure rate μ_{ij} ($i \in |I|$, $j \in |J_i|$), the coefficient variation c_v and the penalty rate c^p. The values of these five factors for this full factorial test bed can be found in Table 4. Each factor has three levels, so in total 243 instances are generated by taking all combinations of the factor levels. We use different D_fs to generate different values of D_0. D_0 is computed based on D_f and the average value of the expected downtimes for two alternative designs of each component, i.e.,

$$D_0 = D_f \sum_{i=1}^{|I|} \frac{1}{|J_i|} \sum_{j=1}^{|J_i|} \mu_{ij} r_{ij} T.$$

Table 4: Parameter setting for the optimization test bed

Name of parameter	No. val.	Values
Downtime budget factor (D_f)	3	1, 1.1, 1.2
Acquisition cost (c^a_{ij})	3	1) $[500, \ldots, 500i, \ldots, 5000]$; $1.5 \times [500, \ldots, 500i, \ldots, 5000]$ 2) $[500, \ldots, 500i, \ldots, 5000]$; $2 \times [500, \ldots, 500i, \ldots, 5000]$ 3) $[500, \ldots, 500i, \ldots, 5000]$; $2.5 \times [500, \ldots, 500i, \ldots, 5000]$
Expected failure rates (μ_{ij})	3	1) $[0.15, 0.14, 0.12, 0.08, 0.06, 0.16, 0.18, 0.2, 0.04, 0.02]$; $[0.1, 0.12, 0.11, 0.06, 0.03, 0.13, 0.14, 0.15, 0.03, 0.01]$ 2) $[0.1, 0.14, 0.12, 0.08, 0.06, 0.16, 0.18, 0.2, 0.04, 0.02]$; $0.75 \times [0.1, 0.12, 0.11, 0.08, 0.03, 0.13, 0.14, 0.15, 0.03, 0.01]$ 3) $[0.1, 0.14, 0.12, 0.08, 0.06, 0.16, 0.18, 0.2, 0.04, 0.02]$; $0.5 \times [0.1, 0.12, 0.11, 0.08, 0.03, 0.13, 0.14, 0.15, 0.03, 0.01]$
Coefficient of variation (c_v)	3	0.3, 0.9, 1.5
Penalty cost factor (c^p)	3	1000, 5000, 10000

In our test bed, the acquisition cost of the second design of every component is more expensive than the first design, so the second design is referred to as the "expensive" design and the first design is referred to as the "cheap" design. Notice that for the expensive design of every component, the average and standard deviation of failure rate are always smaller than the ones for the cheap design, which is sensible since otherwise there's no intention to choose the expensive design.

The constants of the test bed are set as follows: the cost per repair c^r_{ij} ($i \in |I|$, $j \in |J_i|$) is equal to 30% of the corresponding acquisition cost c^a_{ij}. For every instance, we assume that: the life cycle length T is ten years; there are ten components in the system ($|I| = 10$); each component has two alternative designs to choose from ($|J_i| = 2$); the repair time for each failure is 3 hours.

5.2.2. Results comparison of the optimization results between different evaluation methods

For each evaluation method as well as simulation, we generate 243 solutions from the test bed by using full enumeration. As mentioned in Section 5.1.1, for the zero-uncertainty method, $D^E_{A_1}(\boldsymbol{x}) = 0$ for all the instances. For the partial-uncertainty method, c_v are not considered, only 81 instances are computed. To compare the differences among all the 243 parameter settings, we use one value as the optimal policy for different c_v under the same settings of $(D_f, c^a_{ij}, \mu_{ij}, c^p)$.

The results are summarized in Table 5. To compare the LCC from different methods, we

Table 5: Results compare of the optimization results between different evaluation methods

		Zero-uncertainty			Partial-uncertainty			Full-uncertainty		
		$\Delta\pi(\boldsymbol{x}^z)$ (%)	$\Delta P_{\boldsymbol{x}^z}$ (%)	$\Delta P_e(\boldsymbol{x}^z)$ (%)	$\Delta\pi(\boldsymbol{x}^p)$ (%)	$\Delta P_{\boldsymbol{x}^p}$ (%)	$\Delta P_e(\boldsymbol{x}^p)$ (%)	$\Delta\pi(\boldsymbol{x}^f)$ (%)	$\Delta P_{\boldsymbol{x}^f}$ (%)	$\Delta P_e(\boldsymbol{x}^f)$ (%)
D_f	1	105.7	100	0.71	0.31	34.6	0.04	0.02	13.6	0.01
	1.1	83.9	100	0.61	1.32	54.3	0.08	0.02	12.3	0.01
	1.2	60.1	100	0.49	3.21	64.2	0.11	0.03	14.8	0.02
c_f^a	1.5	110.3	100	0.73	1.09	45.7	0.07	0.03	17.3	0.02
	2	79.2	100	0.59	1.63	53.1	0.08	0.02	9.9	0.01
	2.5	60.2	100	0.48	2.12	54.3	0.08	0.02	13.6	0.01
μ_f	1	21.9	100	0.47	1.21	49.4	0.09	0.06	28.4	0.03
	0.75	72.5	100	0.65	1.66	46.9	0.08	0.02	11.1	0.01
	0.5	155.3	100	0.68	1.97	56.8	0.07	0.00	1.2	0.00
c^p	1000	31.8	100	0.45	0.96	50.6	0.07	0.03	17.3	0.02
	5000	72.4	100	0.61	1.81	56.8	0.09	0.03	13.6	0.01
	10000	145.6	100	0.74	2.07	45.7	0.08	0.02	9.9	0.01
c_v	0.3	80.9	100	0.54	0.04	14.8	0.01	0.01	4.9	0.00
	0.9	84.9	100	0.61	1.42	67.9	0.09	0.01	14.8	0.01
	1.5	83.9	100	0.65	3.38	70.4	0.13	0.05	21.0	0.02
All		83.2	100	0.60	1.61	51.0	0.08	0.02	13.6	0.01

first compute the actual LCC by evaluating the optimal policy of different methods ($\boldsymbol{x}^z$, $\boldsymbol{x}^p$ and $\boldsymbol{x}^f$), defined as $\pi(\boldsymbol{x}^z)$, $\pi(\boldsymbol{x}^p)$ and $\pi(\boldsymbol{x}^f)$. Then we compute the gap between $\pi(\boldsymbol{x}^z)$, $\pi(\boldsymbol{x}^p)$, $\pi(\boldsymbol{x}^f)$ and $\pi(\boldsymbol{x}^s)$ which are shown as $\Delta\pi(\boldsymbol{x}^z)$, $\Delta\pi(\boldsymbol{x}^p)$ and $\Delta\pi(\boldsymbol{x}^f)$. We also interest in the gap of the percentage of the optimal solutions different from $\boldsymbol{x}^s$ over the optimal solutions of all the instances, defined as $P_e(\boldsymbol{x}^z)$, $P_e(\boldsymbol{x}^p)$ and $P_e(\boldsymbol{x}^f)$, between the three methods and simulation, $\Delta P_e(\boldsymbol{x}^z)$, $\Delta P_e(\boldsymbol{x}^p)$ and $\Delta P_e(\boldsymbol{x}^f)$. The gap of the percentage of expensive design chosen in the optimal between the three methods and simulation, are also computed as $\Delta P_{\boldsymbol{x}^z}$, $\Delta P_{\boldsymbol{x}^p}$ and $\Delta P_{\boldsymbol{x}^f}$. The full-uncertainty method has the best performance in terms of smallest $\pi(\boldsymbol{x}^z)$, $\pi(\boldsymbol{x}^z)$ and $\Delta\pi(\boldsymbol{x}^z)$. While the zero-uncertainty method has the worst performance. This is because $D^E_{A_1}(\boldsymbol{x}^z) = 0$, which means $P(\boldsymbol{x}^z) = 0$, so for all the instances, $P_e(\boldsymbol{x}^z) = 0$. This results in $\Delta P_e(\boldsymbol{x}^z) = 100\%$ and large values of $\Delta\pi(\boldsymbol{x}^z)$ and $\Delta P_{\boldsymbol{x}^z}$. For the partial-uncertainty method, given that it did not consider the variation of failure rates, it suffers from larger gaps when c_v is large and μ_f is small. Because for smaller μ_{ij}, different c_v can cause big difference in λ_{ij}. With large c_f^a, partial-uncertainty method tend to chose more cheap designs due to underrated $P(\boldsymbol{x})$ comparing to simulation. When D_f increases, each gap of the partial-uncertainty method and full-uncertainty method becomes larger. Because for large D_f, $D^E(\boldsymbol{x})$ becomes very small, which makes it very hard to be approximated.

Generally speaking, full-uncertainty method performs very well under all parameter setting.

5.2.3. Results and managerial insights

Given that we have already proved that the full-uncertainty method has a high accuracy. In this subsection, we focus on discussing the managerial insights of $\boldsymbol{x}^f$ under different parameter settings. We summarize the results of the full-uncertainty method in Table 6 and Table 7. The optimal solution $\boldsymbol{x}^f$ for each instance in the test bed is the choices of alternative designs for ten components. To better demonstrate the solutions we compute the percentage of expensive components chosen in the solutions, denoted as $P_{\boldsymbol{x}^f}$ in Table 6. The approximated probability of $D(\boldsymbol{x}^f)$ exceeding D_0 is calculated for each instance as well, shown as P_{D_0}. $D_E(\boldsymbol{x}^f)$ and the percentage of $D_E(\boldsymbol{x}^f)$ over D_0 are also given. In Table 6, we categorize all the instances containing a specific level of a factor into a subset. For example, value 70.4% in the first row is the average value of $P_{\boldsymbol{x}^f}$ of 81 instances with $D_f = 1$.

To observe the changes of the objective function under the optimal solution, we calculate $\pi(\boldsymbol{x}^f)$, $A(\boldsymbol{x}^f)$, $R(\boldsymbol{x}^f)$ and $P(\boldsymbol{x}^f)$ of each instance and compare them with the ones of a standard instance in which the optimal solution is $\boldsymbol{x}^*$, and all factors $(D_f, c_{ij}^a, \mu_{ij}, c_p, c_2)$ are set at the medium levels that are given in Table 4. For this standard instance, the values of $\pi(\boldsymbol{x}^*)$, $A(\boldsymbol{x}^*)$, $R(\boldsymbol{x}^*)$ and $P(\boldsymbol{x}^*)$ are 64666, 43000, 9998 and 11669 respectively. For each instance, the percentages of ($\pi(\boldsymbol{x}^f)$, $A(\boldsymbol{x}^f)$, $R(\boldsymbol{x}^f)$ and $P(\boldsymbol{x}^f)$ divided by the corresponding values of the standard instance are summarized in Table 7.

The first observation from Table 7 and Table 6 is that when D_0 in the service contract becomes larger (when D_f increases), it is more beneficial to choose the cheap designs, since the expected excess downtime (in terms of P_{D_0}, $\dfrac{D_E(\boldsymbol{x}^f)}{D_0}$ and $D_E(\boldsymbol{x}^f)$) becomes smaller and the penalty cost has less influence on the total expected cost. For similar reasons, when the penalty cost rate c^p increases, choosing more expensive designs becomes more attractive, since in this way the excess downtime can be reduced to compensate the increase in c^p. In contrast, if the prices for the expensive designs increase, we should choose more cheap designs regardless of the fact that the expected excess downtime, the penalty cost and the repair cost will increase by doing this.

Remember that for the expensive design of every component, the average and standard deviation of failure rate are always smaller than the ones for the cheap design. Namely, the reliability performances of expensive designs are better than cheap designs. When the improvement of reliability performances by choosing expensive designs becomes bigger, i.e., μ_{ij}s are set at a higher level given in Table 4, it is more advantageous to have more expensive

designs in the system. This will apparently decrease the excess downtime during the life cycle (in terms of P_{D_0}, $\frac{D_E(\boldsymbol{x}^f)}{D_0}$ and $D_E(\boldsymbol{x}^f)$), together with the penalty cost. Moreover, by choosing more expensive designs the repair cost decreases since the better reliability performances can be obtained.

As we expected, the changes of the standard deviations of failure rates do not have a significant impact on the acquisition cost and repair cost, whereas the penalty cost varies considerably. Since the uncertainties in failure rates will increase the expected excess downtime, we tend to select more expensive designs to avoid the big increment of the expected excess downtime and penalty cost when the uncertainty is large or c_v is large, although the acquisition cost and repair cost will slightly increase.

Another observation based on the maximum and minimum value of $P_{\boldsymbol{x}^f}$ and $P(\boldsymbol{x}^f)$ is that the optimal solution ($P_{\boldsymbol{x}^f}$) and the penalty cost varies a lot among different parameter settings except the cases with $D_f = 1$. This shows that the optimal solution is quite sensitive to the changes of all the parameters and the penalty cost plays an important role in the minimization of expected total cost.

Table 6: Numerical results of the optimization test bed

		$P_{\boldsymbol{x}^f}$ (%)			P_{D_0} (%)			$D_E(\boldsymbol{x}^f)$ (hrs)			$D_E(\boldsymbol{x}^f)/D_0$ (%)		
		Avg.	max	min	Avg.	max	min	Avg.	max	min	Avg.	max	min
D_f	1	70.37	90.00	70.37	25.99	55.62	25.99	2.62	8.60	2.62	9.21	28.26	9.21
	1.2	61.11	90.00	10.00	17.76	38.76	3.52	1.78	6.49	0.16	5.22	17.75	0.55
	1.4	50.49	80.00	10.00	12.71	29.21	1.98	1.28	4.91	0.09	3.24	12.92	0.27
c_f^a	Level 1	73.83	90.00	20.00	15.12	35.36	1.98	1.44	5.42	0.09	4.52	17.81	0.27
	Level 2	59.14	80.00	10.00	18.82	49.79	3.07	1.88	7.54	0.14	5.83	24.76	0.43
	Level 3	49.01	80.00	10.00	22.52	55.62	5.34	2.36	8.60	0.34	7.32	28.26	1.00
μ_f	Level 1	48.52	90.00	10.00	27.30	55.62	6.74	3.20	8.60	0.41	9.33	28.26	0.95
	Level 2	65.68	90.00	20.00	18.57	44.77	3.29	1.73	4.91	0.17	5.59	14.52	0.44
	Level 3	67.78	90.00	30.00	10.59	28.46	1.98	0.75	2.22	0.09	2.74	9.30	0.27
c^p	5000	45.80	80.00	10.00	23.47	55.62	3.07	2.49	8.60	0.14	7.74	28.26	0.43
	10000	61.85	80.00	10.00	17.95	43.80	3.07	1.76	6.28	0.14	5.45	20.63	0.43
	20000	74.32	90.00	20.00	15.04	34.79	1.98	1.43	5.25	0.09	4.48	17.26	0.27
c_v	0.3	53.83	90.00	10.00	17.92	55.62	1.98	1.27	5.47	0.09	3.99	17.96	0.27
	0.9	60.86	90.00	10.00	18.55	51.35	2.54	1.77	6.93	0.14	5.51	22.75	0.42
	1.5	67.28	90.00	10.00	20.00	46.05	5.08	2.64	8.60	0.40	8.17	28.26	1.21
All		60.66	90.00	10.00	18.82	55.62	1.98	1.89	8.60	0.09	5.89	28.26	0.27

Table 7: Summary of the optimization test bed

		$\pi(\boldsymbol{x}^f)/\pi(\boldsymbol{x}^*)$ (%)			$A(\boldsymbol{x}^f)/A(\boldsymbol{x}^*)$ (%)			$R(\boldsymbol{x}^f)/R(\boldsymbol{x}^*)$ (%)			$P(\boldsymbol{x}^f)/P(\boldsymbol{x}^*)$ (%)		
		avg.	max	min	avg.	max	min	avg.	max	min	avg.	max	min
D_f	1	123	271	68	99	127	66	93	159	51	240	901	19
	1.2	103	216	65	92	127	65	90	152	54	154	619	17
	1.4	90	177	63	85	127	65	87	148	54	108	432	6
c_f^a	Level 1	92	234	63	84	90	66	76	100	51	138	879	6
	Level 2	106	253	67	94	106	65	92	129	69	165	901	25
	Level 3	117	271	67	99	127	66	102	159	84	198	901	55
μ_f	Level 1	124	271	66	84	127	65	105	159	88	289	901	49
	Level 2	105	197	67	95	127	67	95	122	75	149	521	26
	Level 3	87	137	63	97	127	74	70	88	51	63	223	6
c^p	5000	87	127	63	82	111	65	85	120	54	107	369	6
	10000	103	187	64	93	127	65	90	133	54	151	538	12
	20000	126	271	66	102	127	69	96	159	51	244	901	15
c_v	0.3	91	182	63	87	127	65	88	152	54	106	432	6
	0.9	103	219	65	92	127	65	90	152	51	154	633	14
	1.5	122	271	67	97	127	66	92	159	51	241	901	23
All		105	271	63	92	127	65	90	159	51	167	901	6

6. Conclusion

In this paper, we introduced a decision support model for a series system with uncertain component reliability and multiple designs for each component during the design phase. We formulated the costs that are affected by the uncertain component reliability and different design for each component throughout the life time of a systems (LCC). Two levels of uncertainty are investigated, the uncertainty of the number of failures of the system and the uncertainty of the failure rates. We proposed three different methods to evaluation a given policy, the zero-uncertainty method does not consider either type of the uncertainty, the partial-uncertainty method only considers the uncertainty of the number of failures in the system, and the full-uncertainty method takes both uncertainty in to account. We conduct two numerical experiment to investigate the accuracy of each method by comparing the results of each method with the simulation. For the evaluation test bed and optimization test bed, the full-uncertainty method performs the best, while the zero-uncertainty method leads to large gap in excess downtime and LCCs. Partial-uncertainty methods also has relatively larger gaps. The results of the optimization test bed also reveals some managerial insights which may benefit the decision making process in practice.

Given that the optimization problem is formulated as a non-linear, non-separable, integer programming problem. It is NP-hard. For optimization, we only use full enumeration which

is time consuming. For evaluation, we only approximate a given policy numerically. In our further research, we focus on getting the analytical results on which design should be chosen to get the optimal solution exactly.

Appendices

A. Procedures of the Monte Carlo simulation

For the numerical experiment of evaluation, we use the following steps:

Step 1 First, we generate the sequences of r_i, μ_i and σ_i, $i = \{1, 2, ..., n\}$. Then we get D_0 immediately from (29). Furthermore, we take one sample $\hat{\Lambda}_i$ from $\Lambda_i \sim G(\mu_i, \sigma_i)$ for each component to simulate its failure rate λ_i, where $G(\mu_i, \sigma_i)$ is a general distribution with parameter mean μ_i and standard deviation σ_i. Given that the number of failures of each component s_i is Poisson distributed with parameter $\lambda_i T$, we take one sample $\hat{S}_i$ from $S_i \sim$Pois$(\hat{\Lambda}_i T)$. Together with r_i, we get one simulation result $\hat{D}_S^E/D_0$ of the proportion of D^E/D_0 computed as following:

$$\hat{D}_S^E/D_0 = \frac{\sum_{i=1}^n \hat{S}_i r_i - D_0}{\sum_{i=1}^n \hat{S}_i r_i}. \tag{30}$$

Step 2 Repeat step 1 for 10000 times to get 10000 $\hat{D}_S^E/D_0$, we take the expected value $\bar{D}_S^E/D_0$ of all the $\hat{D}_S^E/D_0$ and compare it with the approximation result D_{A1}^E/D_0, D_{A2}^E/D_0 and D_{A3}^E/D_0 to get values of the absolute gap between $\bar{D}_S^E/D_0$ and D_{A1}^E/D_0, D_{A2}^E/D_0 and D_{A3}^E/D_0.

Step 3 Repeat step 3 for 50 times to generate final simulation results of D^E/D_0, D_S^E/D_0. And the mean and maximum value for each gap and the confidence interval for $D_S^{E_p}$. The 95% percent confidence interval is given as:

$$(D_S^E/D_0 - t_{(49,2.5\%)}\sqrt{\frac{S^2(50)}{50}}, \quad D_S^E/D_0 + t_{(49,2.5\%)}\sqrt{\frac{S^2(50)}{50}}).$$

For the numerical experiment of optimization, the simulation procedure is similar, for each feasible solution of a certain instance, we get $A(\boldsymbol{x})$ and $R(\boldsymbol{x})$ from (1) and (2). To compute $P(\boldsymbol{x})$, we first generate 10000 samples of $\hat{\Lambda}_{ij}$ ($i \in \{1, 2, \ldots, 10\}$, $j \in \{1, 2\}$), from

$\Lambda_{ij} \sim G(\mu_{ij})$ and 10000 samples of $\hat{S}_{ij}$ ($i \in \{1, 2, \ldots, 10\}$, $j \in \{1, 2\}$) from $S_i \sim Pois(\hat{\Lambda}_i T)$ respectively. According to (3), we get 10000 values of $\hat{P}(\boldsymbol{x})$, and take the expected value $\bar{P}(\boldsymbol{x})$. Run this procedure 50 times to general 50 $\bar{P}(\boldsymbol{x})$, use the expected value as the final evaluation value, $P(\boldsymbol{x})$. Find the feasible solution with the minimum $\pi(\boldsymbol{x})$ by solving problem (P) with full enumeration.

B. Derivations for the full-uncertainty method

B.1. Identities for the Erlang$(k-1, k)$ distribution

Consider an Erlang$(k-1, k)$ distribution X with parameters (k, θ, q), the probability density function is

$$e_{k-1,k}(x) = q\theta^{k-1}\frac{x^{k-2}}{(k-2)!}e^{-\theta x} + (1-q)\theta^{k}\frac{x^{k-1}}{(k-1)!}e^{-\theta x},$$

and the cumulative distribution function of X is

$$E_{k-1,k}(x) = q\left(1 - \sum_{j=0}^{k-2}\frac{(\theta x)^j}{j!}e^{-\theta x}\right) + (1-q)\left(1 - \sum_{j=0}^{k-1}\frac{(\theta x)^j}{j!}e^{-\theta x}\right).$$

Then the first partial moment of the Erlang$(k-1, k)$ can be described as:

$$\begin{aligned}
\mathbb{E}[(X - X_0)^+] &= \int_0^\infty (x - X_0)^+ e_{k-1,k}(x)dx \\
&= \frac{q(k-1)}{\theta}\int_{X_0}^\infty \theta^k \frac{x^{k-1}}{(k-1)!}e^{-\theta x}dx + \frac{k(1-q)}{\theta}\int_{X_0}^\infty \theta^{k+1}\frac{x^k}{k!}e^{-\theta x}dx \\
&\quad -X_0\int_{X_0}^\infty e_{k-1,k}(x)dx.
\end{aligned}$$

Given the probability density function of an Erlang distribution is: $e_k^\theta(x) = \theta^k \frac{x^{k-1}}{(k-1)!}e^{-\theta x}$ and the Erlang cumulative distribution function is $E_k^\theta(X) = 1 - \sum_{j=0}^{k-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x}$, we have:

$$\begin{aligned}
\mathbb{E}[(X - X_0)^+] &= \frac{q(k-1)}{\theta}\left[1 - E_k^\theta(X_0)\right] + \frac{k(1-q)}{\theta}\left[1 - E_{k+1}^\theta(X_0)\right] - X_0\left[1 - E_{k-1,k}^\theta(X_0)\right] \\
&= (\frac{k-q}{\theta} - X_0)\sum_{j=0}^{k-2}\frac{(\theta X_0)^j}{j!}e^{-\theta X_0} + (\frac{k-q}{\theta})\frac{(\theta X_0)^{k-1}}{(k-1)!}e^{-\theta X_0}.
\end{aligned}$$

B.2. Identities for the Hyperexponential distribution

For a Hyperexponential distribution X with parameters (θ_1, θ_2, q), the probability density function is given as:

$$h_2(x) = q\theta_1 e^{-\theta_1 x} + (1-q)\theta_2 e^{-\theta_2 x},$$

the cumulative distribution function is given as:

$$H_2(x) = q(1 - e^{-\theta_1 x}) + (1-q)(1 - e^{-\theta_2 x}),$$

and the first partial moment is given as:

$$\begin{aligned}
\mathbb{E}[(X - X_0)^+] &= \int_0^\infty (x - X_0)^+ h_2(x) dx \\
&= \int_{X_0}^\infty x q\theta_1 e^{-\theta_1 x} dx + \int_{X_0}^\infty x(1-q)\theta_2 e^{-\theta_2 x} dx - X_0 \int_{X_0}^\infty h_2(x) dx \\
&= qX_0 e^{-\theta_1 X_0} + \frac{q}{\theta_1} \int_{X_0}^\infty \theta e^{-\theta_1 x} dx + (1-q)X_0 e^{-\theta_2 X_0} + \frac{1-q}{\theta_2} \int_{X_0}^\infty \theta_2 e^{-\theta_2 x} dx \\
&\quad -X_0 \Big[1 - H_2(X_0)\Big].
\end{aligned}$$

With the exponential probability density function $f_\theta(x) = \theta e^{-\theta x}$, and the exponential cumulative distribution function $F_\theta(x) = 1 - e^{-\theta x}$, denote the complementary distribution function of $F_\theta(x)$ as $\bar{F}_\theta(x)$, then the first partial moment can be written as:

$$\begin{aligned}
\mathbb{E}[(X - X_0)^+] &= qX_0 e^{-\theta_1 X_0} + (1-q)X_0 e^{-\theta_2 X_0} + \frac{q}{\theta_1}\Big[1 - F_{\theta_1}(X_0)\Big] + \frac{1-q}{\theta_2}\Big[1 - F_{\theta_2}(X_0)\Big] \\
&\quad -X_0 \Big[1 - H_2(X_0)\Big].
\end{aligned}$$

7. References

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