SPECTRA

Final report by Balint Virag

"Colors blind the eye, sounds deafen the ear" says Lao-Tze, a Chinese philosopher of the 6th century BC. We rely on spectral phenomena, such as colors, sounds, waves, phone and radio signals every day. The goal of this project was to study mathematical aspects of spectra and its relationship to randomness.

We are studying simple but essential mathematical models of real world objects. Random matrices were introduced by Wigner in the 50s to model the nuclei for large atoms. The study of Random Schrodinger operators was initiated by by Anderson in the 50s to model conductance. Random graphs were first popularized by Erdos and Renyi, and they can be a model for certain real-world networks. All these objects have interesting spectral properties that are at the forefront of current mathematical research. Our project set out to tackle several interesting questions in this area.

Random Schrodinger operators in one dimension

One could model a long wire by an infinite discrete path. The wave functions corresponding to these path are extended to infinity, and this essentially shows that it would be a superconductor. But thin wires don't conduct well; a better model is a randomly perturbed version. Here the wave functions are localized, which suggests that indeed, they are insulators. We studied three aspects of these models with three students.

First, with Ben Rifkind, we studied what the shape of the localized wave functions look like. It turns out that they follow a shape which is the exponential of a Brownian motion minus the absolute value function. This is an interesting random distribution with surprising properties.

Second, with Eric Hart, we studied the regularity of the average spectrum. This is measured by a so-called Holder exponent, a number between 0 and 1; we shoed that the Holder exponent tends to 1 very fast as the randomness decreases.

Third, with Marcin Kotowski, in a version of this model studied by Freeman Dyson in the 50s, we proved that at energy zero there is a logarithmic spike. This has to do with a certain symmetry of the model near the energy zero.

Random graphs

A sparse random regular graph is a large network where every node has the same fixed number of connections. This model of random networks is often used to theoretically test algorithms on real-world network. Random local algorithms are a class where one puts a random value at every node, and then nodes pick their state according to what they see in a finite neighborhood. This can be used to create a proper coloring, or an independent set in a graph, for example.

With Viktor Harangi we studied local algorithms that use the wave functions on the graph to create independent sets. These algorithms broke the record of set by previous greedy algorithms designed for this purpose.

With Mutazee Rahman we showed that these algorithms when the connectivity degree is large, these algorithms cannot do very well. More precisely, they can only find independent sets whose size are half the optimal value. These ideas were carried further by my student Mustazee Rahman in a followup paper.

With Agnes Backhausz we studied the correlation structure of processes of these local algorithms. We were able to give a complete characterization of what correlation structures arise this way. Our results extend to more complex graphs, such as Cayley graphs of groups. These are generalizations of the Euclidean lattices.

Some of this work is still in preparation, some of it can be found online one the arxiv.

Preprints

Holder continuity of the integrated density of states in the one-dimensional Anderson model. With Eric Hart. arXiv:1506.06385.

Spectral measures of factor of i.i.d. processes on vertex-transitive graphs. With Agnes Backhausz. arXiv:1505.07412.

Local algorithms for independent sets are half-optimal. With Mustazee Rahman. arXiv:1402.0485.

Factor of iid percolation on trees. Mustazee Rahman. arXiv:1410.3745

Invariant Gaussian processes and independent sets on regular graphs of large girth. With Endre Cska, Balzs Gerencsr, Viktor Harangi. arXiv:1305.3977.